Dynamic input allocation

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Relatore: Prof. Sergio Galeani
to my father
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Chapter 1

Introduction

In modern control systems we have often to deal with plants characterized by the presence of multiple actuators. In many cases the number of these actuators can be even larger than the number of the output quantities we want to control; sometimes the plants are purposely designed with this intrinsic redundancy for various reasons: in order to have actuators with different dynamic characteristics, like available magnitude or rate ranges, or to be safer with respect to possible failures.

In such cases it can be useful to divide the control design process into two steps. In the first step a controller is designed assuming that a virtual control input can be defined, which has the same dimensionality of the controlled output. This virtual control input can consist in many cases simply in a selection of the available control inputs. In a second step, a second control block is designed with the task of choosing a smart way to distribute this control effort between all the available control inputs. In fact, in general, there are infinite different combinations of the inputs having the same effect of the chosen virtual input. This second control block is usually called in the literature an input allocator (or control allocator).

Many different criterions can be used in designing the allocator,
also depending on what we consider a “smart” choice. A possible optimization task for the allocator, which mainly is addressed in this thesis, is that of keeping the control inputs inside the actually available ranges, i.e. far from their saturation values. This objective is relevant for many reasons. First of all because often the controller, given a-priori or designed in a first step, is linear, designed considering a linear model of the plant and so without taking into account the input saturation nonlinearities, which are instead always present. This can bring to performance loss or even to instability. A possible solution to deal with the saturations problem is the use of anti-windup techniques. anyway anti-windup can solve the problem, roughly speaking, only if the control inputs chosen by the linear controller go just temporarily out of the range. For this reason a dynamic allocator which tries to keep the input signals inside their range, or also just slowly brings them back inside, can be used together with an anti-windup to solve the problem.

Moreover, even if the input allocation problem is usually dealt with when the plant is over-actuated, input allocation techniques actually come to be useful also in the case of square or even under-actuated plants. Differently from the case of over-actuated plants, in which the input allocation does not change (at least at the steady state) the controlled output, in this more general case any changes in the control input results in a deterioration of the controlled output. anyway in some situations changing the input signal can be worth some performance and precision losses. For this reason in this thesis the input allocation is generalized to under-actuated plants in such a way as to reach a trade-off between the two contrasting objectives of choosing a “better” input and maintaining the output as closed as possible to the desired value.

This thesis is structured as follows. In Part I the main theoretical results obtained during my PhD course are presented.
In Chap. 3, in particular, the dynamic control allocation problem for input-redundant linear systems is addressed, distinguishing between the strong and weak redundancy cases.

In Chap. 4 the results presented in Chap. 3 are generalized to the case of non input-redundant systems and to the optimization of more general cost functions.

In Chap. 5 the dynamic input allocation approach is adapted to the output regulation problem, in order to deal with possible sinusoidal references or disturbances.

In Part II some applications in the field of nuclear fusion are presented. In particular, some of the dynamic allocation techniques proposed are used to deal with plasma control problems in tokamaks.

In Chap. 7, in particular, the dynamic allocator presented in Chap. 3 is adapted to obtain an elongation regulator for the FTU tokamak’s plasma position control system.

In Chap. 8, the dynamic allocation approach for non input-redundant systems presented in Chap. 4 is used to integrate a saturation avoidance module for the plasma shape controller of the JET tokamak.

The research activity carried out during the PhD studies and discussed in this thesis produced the following publications in international journals and international conferences:


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Part I

Dynamic Input Allocation
Traditionally, the problem of input allocation (also referred to as control allocation) arises when the plant to be controlled is over-actuated. By this we mean that the actuators available for the particular control task we have to deal with are more than the quantities we actually need to control. This can happen in different situations and for different reasons. For example in some cases the available actuators are not powerful enough for the task and so many actuators have to be used in parallel. Sometimes actuators with different dynamic characteristics must be used together in order to overcome the limitations of each one of them. This happens, for example, in dual stage actuator control for hard disk drives where a fast and small actuator is connected in parallel to a slow and large actuator. Sometimes the control effort is divided between many actuators having the same effect on the systems just to be less stressed and overloaded, so that they can remain reliable in the long run. Also it can be done to make the control system more fault tolerant.

In each of these cases we have to face the problem of deciding how to distribute the control effort between the actuators. Many different ways exist to face the problem (see [1, 2] for alternatives). In the input allocation approach the controller design is divided into two (possibly
decoupled) steps. First a control which ensures desired performance is designed. Then a second block is designed to choose the “best” (with respect to some optimization criterion) input, between the infinite equivalent, which still reaches the same performance. This second block is usually called “allocator”.

In the literature control allocation problems have been mostly addressed with reference to specific applications. The most studied one is probably reconfigurable flight control where several techniques have been proposed (see [3] for a comprehensive survey). Another relevant field, where a simpler control allocation problem has been addressed in different ways, is that of dual stage actuator control in hard disk drives (see [4] for a survey). Many works also have been published on control allocation applied to the field of ships and underwater vehicles (see [5] for a survey). New examples of emerging control applications where control allocation is relevant are the novel actuator devices recently proposed in [6]. Another important area where input allocation is needed is that of nuclear fusion reactors, like tokamaks or RFPs. In this context, the demand for fast and efficient input allocators will become more and more relevant as superconducting coils (associated with severe rate limitations) will be adopted in conjunction with conducting coils (characterized by severe magnitude limitations).

Even though the literature in these fields is quite disconnected, in the last decade there has been an increasing effort in trying to unify the different approaches (see, e.g., the invited session [7]). Most of the allocation techniques adopted in practice correspond to static allocators which aim at optimizing certain criteria on the allocated input (see, for example Harkegard and Glad (2005) and references therein).

For the aerospace applications, [8] recognizes the strong coupling between the control allocation problem and the presence of rate and magnitude saturation limits characterizing each actuator. Also [9]
proposes allocation techniques for nonlinear actuators, while in [10] efficient algorithms are proposed for the on-line computation of the optimal static allocation. In [11] spacecraft thruster configuration is addressed based on efficient optimization methods. In [12] actuator dynamics are accounted for in the allocation scheme, which is a Receding Horizon law (the solution to a static optimization problem).

In the majority of the cited works static techniques are used for input allocation. Anyway some dynamic approaches have also been proposed, which are more similar to the key ideas in this thesis. In [13] a dynamic approach is proposed where the dynamical allocation asymptotically recovers the pseudo-inverse allocation for redundant linear systems. The approach is therein formulated for nonlinear systems and also applied to over-actuated ship maneuvering.

In [14] a nonlinear dynamic allocation scheme was proposed to address both cases where the plant input manipulation can be arbitrarily fast (strong redundancy) and cases where it should be sufficiently slow (weak redundancy). That work explicitly focuses on the presence of (rate and magnitude) actuator saturations and optimize the allocation in view of those saturations.

In these works dynamics is used inside the allocator equations. The core idea of these papers is that when strong input redundancy (see Ch. 3 for a precise definition) is at hand, it is possible to inject an arbitrary signal in certain input directions without affecting the state response of the plant. For linear systems (or nonlinear systems which are affine in the control) this means that the redundancy (in the strong sense) lies in the null-space of the input matrix. Anyway also the case of weak redundancy exists in which directions exist which do not affect just the steady-state output response of the plant.

Another research direction is that of control allocation in the presence of disturbances. Such a problem can be dealt with in an output regulation perspective. To the best of our knowledge, the output regulation problem for linear over-actuated systems, with an explicit exploitation
of the control input redundancy, has been investigated first in [15] in the context of tracking control for a linearized model of a hypersonic aircraft, and later extended to encompass linear parameter-varying systems [16]. The steady-state optimization for an input-redundant linear system with nonlinear output function has been considered in [17], with exosystem model restricted to pure integrators. For the same type of exosystem, the results in [14, 18] provide a framework allowing for nonlinear dynamic allocation solutions.

As we will see in this thesis another research direction is that of generalizing the input allocation problem also to plants that are not over-actuated.

In this part of the thesis the main theoretical results obtained in our research during the PhD school are presented.

In Ch. 3, in particular, the dynamic control allocation problem for input-redundant linear systems is addressed, distinguishing between the strong and weak redundancy cases.

In Ch. 4 the results presented in Ch. 3 are generalized to the case of non input-redundant systems and to the optimization of more general cost functions.

In Ch. 5 the dynamic input allocation approach is adapted to the output regulation problem, in order to deal with possible sinusoidal references or disturbances.
Chapter 3

Dynamic input allocation for input-redundant systems

3.1 Introduction

In this chapter the ideas first appeared in [14] are briefly summarized in order to better understand the subject of the next chapters. In particular dynamic control allocation problem for input-redundant linear systems is addressed, distinguishing between the strong and weak redundancy cases.
3.2 Strongly input-redundant systems

In the following we will always consider a linear control system, in which the plant can be described by equations

\[\begin{align*}
\dot{x} &= Ax + Bu + B_d d, \\
y &= Cx + Du + D_d d,
\end{align*}\]

where \(x \in \mathbb{R}^n\) is the state variables vector, \(u \in \mathbb{R}^m\) is the control inputs vector, \(d \in \mathbb{R}^q\) is the exogenous disturbances vector, \(y \in \mathbb{R}^p\) is the vector of the measured outputs, available for a feedback control and \(A, B, B_d, C, D, D_d\) are real matrices of suitable dimension. We suppose to have a given linear controller

\[\begin{align*}
\dot{x}_c &= A_c x_c + B_c u_c + B_r r, \\
y_c &= C_c x_c + D_c u_c + D_r r,
\end{align*}\]

where \(x_c \in \mathbb{R}^{n_c}\), \(u_c \in \mathbb{R}^p\) and \(y_c \in \mathbb{R}^m\) represent, respectively, the controller’s state, input and output vectors while \(r \in \mathbb{R}^{n_r}\) is an exogenous reference signal, connected to the plant (3.1) (as in Fig. 3.1) through

\[\begin{align*}
u_c &= y, \\
u &= y_c.
\end{align*}\]

We will refer to the control system described by (3.1),(3.2) and (3.3) as the original closed-loop system. The original closed-loop system is said to be well-posed if, for every initial condition of the two subsystems \(x(0) = x_0\) and \(x_c(0) = x_{c0}\), the output signals \(y\) and \(y_c\) exist and are unique. It can be proved that this condition corresponds to impose that the matrix \(I - DD_c\) (or, equivalently, the matrix \(I - D_c D\)) is non singular (see e.g. [19, Ch. 3]). When the closed loop is well-posed it
3.2 Strongly input-redundant systems

3.2.1 System model

can be written in the form

$$\dot{x}_{cl} = A_{cl}x_{cl} + B_{cl}w, \quad (3.4a)$$
$$y_{c} = C_{cl,y_{c}}x_{cl} + D_{cl,y_{c}}w, \quad (3.4b)$$
$$y = C_{cl,y}x_{cl} + D_{cl,y}w. \quad (3.4c)$$

with state $x_{cl} = [x^T \ x_c^T]^T$ and exogenous input $w = [r^T \ d^T]^T$ and where the matrices $A_{cl}, B_{cl}, C_{cl,y_{c}}, D_{cl,y_{c}}C_{cl,y}, D_{cl,y}$ can be easily computed from the matrices of the two subsystems (3.1) and (3.2). Reminding that a system of the form (3.1) is said to be internally stable if its state matrix $A$ is Hurwitz, i.e has all eigenvalues with negative real part, we suppose that the controller (3.2) is designed in such a way that the following assumption is satisfied.

**Assumption 3.1** The original closed-loop system is well-posed and internally stable.

![Block diagram of the original closed-loop system](image)

Figure 3.1: Block diagram of the original closed-loop system. The block $\mathcal{P}$ represents the plant (3.1), while the block $C$ represents the given controller (3.2).

In this chapter we are interested in plants which have an input redundancy, i.e. in which, roughly speaking, there are more control inputs than controlled outputs. To be more precise we can distinguish two different concepts of input redundancy, a stronger one and a weaker one. We formalize the strong redundancy with the following definition.
Definition 3.1  The plant (3.1) is strongly input redundant from $u$ to $y$ if it satisfies

$$\ker\left(\begin{bmatrix} B \\ D \end{bmatrix}\right) \neq \{0\}.$$  \hspace{1cm} (3.5)

When the plant is strongly input redundant we can define a full column-rank matrix $B_{\perp}$ such that

$$\text{Im}(B_{\perp}) = \ker\left(\begin{bmatrix} B \\ D \end{bmatrix}\right).$$  \hspace{1cm} (3.6)

The matrix $B_{\perp}$ just defined represents a basis for a non trivial subspace of the input space $\mathbb{R}^m$ whose elements are control inputs which affect neither the state nor the outputs of the plant at all. Since these input vectors have no effect on the plant, they can be exploited to change the total input and choose, between many equivalent input values, the “best” one with respect to some criterion.

![Block diagram of the input allocated closed-loop system.](image)

The new block $A$ (darker in the figure) represents the input allocator (3.7). The allocator is interposed between the controller and the plant and adds its own contribution $y_a$ to the controller output $y_c$ in order to have a better total control input $u$.

We want to design an additional block which exploits the input redundancy of the plant in order to modify the input decided by the controller in order to have a new one more convenient (in some sense) without changing the resulting output.
A possible idea is to design the new allocator block as
\[
\begin{align*}
\dot{x}_a &= -\rho K B^T \perp W u, & (3.7a) \\
y_a &= B \perp x_a. & (3.7b)
\end{align*}
\]
and to connect it to the closed-loop system (as in Fig. 3.2) through
\[
u = y_c + y_a. \quad (3.8)
\]
We will refer to the control system obtained by the interconnection of (3.1), (3.2), (3.7), (3.3a) and (3.8) as the input allocated closed-loop system.

In the following theorem, whose proof can be found in \cite{14}, the main properties of the input allocated closed-loop are explained.

**Theorem 3.1** Assume that the plant (3.1) is strongly input redundant and the interconnection of (3.1) and (3.2) by (3.3) is well-posed. If $K$ and $W$ in (3.7) are symmetric matrices satisfying $K > 0$ and $B^T \perp W B \perp > 0$, $\rho$ is a positive scalar and $B \perp$ satisfies (3.6), then:

- the input allocated closed-loop system is well-posed if and only if the original closed-loop system is well-posed. Moreover it is internally stable if and only if the original closed-loop system is internally stable;
- given any initial condition $x(0), x_c(0), x_a(0)$ and any selection of the external signals $r(\cdot)$ and $d(\cdot)$, the plant output responses of the two systems coincide for all times.

The allocator action consists in moving the input to a different steady-state value, without perturbing the output and the state of the plant at all. As for the new steady-state input $\bar{u}$, it can be proven (see Ch. 4) that it corresponds to the solution of the following static constrained optimization problem:

\[
\begin{align*}
u^* &= \min_{x_a} J(u) \\
\text{subject to : } u &= y_c + B \perp x_a.
\end{align*}
\]
with the cost function

$$J(u) := \frac{1}{2} u^T W u.$$  \hspace{1cm} (3.10)

From this result, the parameter matrix $W$’s role is evident: it’s a weight matrix. The use of the different inputs is penalized quadratically and we can specify different weights for them by choosing an appropriate value for $W$. In particular, if $W$ is a diagonal matrix with positive entries, then the inputs corresponding to larger weights are more penalized and the allocator action will be directed at using them less than the others.

As for the parameter $K$, it comes out that the steady-state input $\bar{u} = u^*$ does not depend on it. What is influenced by $K$ is just the dynamic behaviour of the allocator, which will converge faster with larger values of $K$. In particular, the convergence speed can be assigned differently to each allocation direction by choosing a diagonal $K$. Moreover, since the results in Theorem 3.1 are true as long as $K$ is positive definite, the allocator dynamics can be chosen as fast as we want. The scalar parameter $\rho$ has the same role of $K$, but it affects all the allocation directions in the same way.

**Remark 3.1** The allocator (3.7) solving dynamically the optimization problem (3.9), chooses the “smallest” (with respect to the weight $W$) input $\bar{u}$ between all the inputs which have the same effect on the plant. A slightly different goal can be to choose the input closer to a desired one $u_r$. This is obtained by substituting (3.7a) with $\dot{x}_a = -KB_TW(u - u_r)$.

### 3.3 Weakly input-redundant systems

In this section we will consider a different concept of input redundancy, weaker than the *strong input redundancy* defined in the previous section, but anyway exploitable for our purpose. In order to
formalize this weak redundancy concept, let introduce the matrix
\[ \bar{P} := \lim_{s \to 0} P(s) \]
where \( P(s) = C(sI - A)^{-1}B + D \) is the transfer matrix of the plant (3.1). The matrix \( \bar{P} \) represents the plant input/output map at the steady-state.

**Definition 3.2** The plant (3.1) is weakly input redundant from \( u \) to \( y \) if it satisfies

\[ \ker (\bar{P}) \neq \{0\}. \]  

(3.11)

When the plant is weakly input redundant we can define a full column-rank matrix \( B_\perp \) such that

\[ \text{Im}(B_\perp) = \ker (\bar{P}). \]

(3.12)

The matrix \( B_\perp \) just defined represents a basis for a non trivial subspace of the input space \( \mathbb{R}^m \) whose elements are control inputs which do not affect the outputs of the plant “at the steady-state” for constant inputs. On the other hand, differently from the case of strong redundancy, these inputs do affect the state evolution and also the output during the transient.

For weakly input redundant systems an allocator with the same structure (3.7) can be designed and interconnected to the original closed-loop in the same way. For the closed-loop the following theorem can be proved (see [14]).

**Theorem 3.2** Assume that the plant (3.1) is weakly input redundant and that the original closed-loop system satisfies Assumption 3.1. If \( K \) and \( W \) in (3.7) are symmetric matrices satisfying \( K > 0 \) and \( B_\perp^TWB_\perp > 0 \) and \( B_\perp \) satisfies (3.12), then there exists a small enough \( \rho > 0 \) such that:

\[ \text{1} \]

We can assume, without loss of generality, that the limit exists finite. If it does not exist and some entries go to infinity, then it means that the corresponding inputs at the steady-state must be equal to zero, in order to have internal stability. For this reason these inputs cannot be allocated to different values and so we’re not interested in them. We can assume that (3.1) refers just to the part of the plant with finite entries in \( \bar{P} \).
• the input allocated closed-loop system is internally stable;

• given any initial condition \(x(0), x_c(0), x_a(0)\) and any converging \(^2\) selection of the external signals \(r(\cdot)\) and \(d(\cdot)\), the plant output responses of the two systems converge to the same values.

In the case of weakly redundant plants, we can see that the same allocator ensures the same results at the steady-state, as long as its dynamics are slow enough with respect to the original closed-loop ones. In fact, while in the case of strong redundancy the two dynamics are completely decoupled, in the case of weak redundancy this is not true. So, to prove stability of the allocated closed-loop, it is convenient to resort to a time scales separation.

**Remark 3.2** Even if such an allocation method needs to be slow, it remains appealing anyway for facing saturation constraints on the actuators, especially if combined with anti-windup techniques to face the saturation nonlinearities.

Such techniques, roughly speaking, allow to augment a given linear control system in order to manage the nonlinear behaviour caused by the saturation nonlinearity and assure the stability of the closed-loop. The anti-windup compensation can work as long as the input signal decided by the linear controller exceeds the available range just temporarily, during the transient. So the use of input allocation should be seen as synergistic with anti-windup techniques, in such a way that the latter account for saturation during transients, whereas the former addresses saturations at the steady-state.

\(^2\)We denote as “converging signal” any function \(t \mapsto s(t)\) such that \(\lim_{t \to \infty} s(t)\) is well defined and finite.
Chapter 4

Dynamic input allocation for non input-redundant systems

4.1 Introduction

We saw in Chap. 3 that when the number of actuators in a Multiple Input Multiple Output (MIMO) system is larger than the number of controlled outputs, many different choices of the input functions can give rise to the same output response (in particular, at least the same steady-state output for constant exogenous signals can be obtained by infinitely many choices of the constant input). In this context, input allocation allows to change the control input without affecting the output (at least asymptotically). One of the weaknesses of looking at input allocation as something completely invisible from the plant output is that this type of problem
setting does not capture a quite common situation in MIMO control systems, where there is no actual redundancy in the available inputs, neither strong nor weak redundancy. This is the generic situation for “square” and “tall” plants (i.e. whenever \( m \leq p \)). In this case, input allocation inevitably affects both the transient and the steady-state output response.

From this broader viewpoint, input allocation can be then better understood as some way to trade some output performance (which could be seen as the steady-state output value induced by a certain input pattern) for a more desirable input allocation (e.g., keeping the actuators away from the saturation limits).

In this chapter we revisit the dynamic allocation scheme for input redundant plants described in Chap. 3 and propose generalizations that apply to cases where the plant under consideration is not input redundant but the control specifications allow to modify the output within certain bounds.

In order to achieve this type of trade-off, we revisit here the approach used in Chap. 3 by introducing a cost function which allows a precise definition of the trade-off, and then we show that the proposed modification of the allocator is able to modify the system input in such a way to minimize the overall cost. As a result, not only we extend the approach of Chap. 3 to the non redundant case, but we also show that, given a possibly nonlinear cost function that satisfies some mild conditions, a suitable selection of the allocator dynamics based on the gradient of that cost function allows to guarantee convergence to the minimum of the cost function, thus performing asymptotically an optimal trade-off.

The results presented in this chapter can be found in [20], [21], [18] and [22].
4.2 Trading input variations for output variations

We consider a plant described by the linear system

\[
\begin{align*}
\dot{x} &= Ax + Bu + B_d d, \\
y &= Cx + Du + D_d d,
\end{align*}
\]  

(4.1a) \hspace{1cm} (4.1b)

where \( x \in \mathbb{R}^n, \ u \in \mathbb{R}^m, \ d \in \mathbb{R}^q, \ y \in \mathbb{R}^p \), controlled by an a-priori given linear controller

\[
\begin{align*}
\dot{x}_c &= A_c x_c + B_c u_c + B_r r, \\
y_c &= C_c x_c + D_c u_c + D_r r,
\end{align*}
\]  

(4.2a) \hspace{1cm} (4.2b)

where \( x_c \in \mathbb{R}^{n_c}, \ u_c \in \mathbb{R}^p, \ y_c \in \mathbb{R}^m, \ r \in \mathbb{R}^{n_r} \), under the interconnection conditions

\[
\begin{align*}
u_c &= y, \\
u &= y_c,
\end{align*}
\]  

(4.3a) \hspace{1cm} (4.3b)

as shown in Fig. 3.1. If the closed loop system (4.1), (4.2), (4.3) is well posed, it is possible to define the closed loop system with state \( x_{cl} = [x^T, x_c^T]^T \) and exogenous input \( w = [r^T, d^T]^T \) as

\[
\begin{align*}
\dot{x}_{cl} &= A_{cl} x_{cl} + B_{cl} w, \\
y_c &= C_{cl,y_c} x_{cl} + D_{cl,y_c} w, \\
y &= C_{cl,y} x_{cl} + D_{cl,y} w.
\end{align*}
\]  

(4.4a) \hspace{1cm} (4.4b) \hspace{1cm} (4.4c)

We will refer to (4.4) as the original closed loop system. We make the following natural assumption.

**Assumption 4.1** The closed loop system (4.4) is well posed and internally stable.

Define the transfer matrix from \( u \) to \( y \) for the plant (4.1) as

\[
P(s) = C(sI - A)^{-1} B + D;
\]  

(4.5)
we assume for simplicity that $\bar{P} := \lim_{s \to 0} P(s)$ exists (see Chap. 3 for more details). If this assumption is satisfied and the exogenous signals are converging signals, we know that at the steady-state all the signals tend to a constant value and the plant output results to be

$$\bar{y} = \bar{P} \bar{u},$$

(4.6)

where the superscript $\bar{\cdot}$ on a signal denotes the constant value it approaches at the steady-state.

The input allocation we propose is realized by inserting an additional subcompensator between the controller output $y_c$ and the plant input $u$, in order to modify the actual input to the plant as

$$u = y_c + \delta u,$$

(4.7)

where $\delta u$ is a suitable signal computed by the allocator with the goal of choosing a “better” input, in some sense to be specified. When such an allocator is inserted in the loop, in the general case of a system without input redundancy, any changes to the control input $u$ made by the allocator will result in some changes on the controlled output $y$. If the whole system is still well posed and asymptotically stable, we have at the steady-state

$$\bar{y} = \bar{P} \bar{u}$$

$$= \bar{P} (\bar{y}_c + \delta \bar{u})$$

(4.8)

$$= \bar{P} \bar{y}_c + \bar{P} \delta \bar{u}$$

(4.9)

thus resulting in an output variation, with respect to the original closed loop output in (4.6), given by

$$\delta \bar{y} = \bar{P} \delta \bar{u}.$$  

(4.10)

Assuming that the original linear controller (4.2) was designed to have a good output response $\bar{y}$, without taking care of any “cost” of the
input required, the allocator goal will be in general to operate a trade-off between the two contrasting needs of finding a better control input $u$ and not changing too much the output $y$.

In particular we can desire to constraint the allocator to act just on some input directions. If we represent these directions through the columns of a full column rank matrix $B_0 \in \mathbb{R}^{m \times n_a}$, with $0 < n_a \leq m$, we can write the allocator contribution as

$$\delta u = B_0 x_a \tag{4.11}$$

where the vector $x_a \in \mathbb{R}^{n_a}$ represents the degrees of freedom left to the allocator.

To better define our goal, let introduce a continuously differentiable cost function $J(\bar{u}, \delta \bar{y})$ measuring the trade-off between the modified steady-state value of the plant input $\bar{u}$ and the associated output modification $\delta \bar{y}$.

At the steady-state the relations $\bar{u} = \bar{y}_c + B_0 \bar{x}_a$ and $\delta \bar{y} = \bar{P}B_0 \bar{x}_a$ hold. So, if we make explicit the dependence of $J(\bar{u}, \delta \bar{y})$ on $\bar{x}_a$ and we consider $\bar{y}_c$ as a fixed parameter (so that dependence on $\bar{y}_c$ will not be indicated), it is possible to define the function

$$\bar{J}(\bar{x}_a) := J(\bar{y}_c + B_0 \bar{x}_a, \bar{P}B_0 \bar{x}_a). \tag{4.12}$$

The allocator block is described by the equations:

$$\dot{x}_a = -\rho K \left( \nabla \bar{J} \right)^T$$

$$= -\rho K B_0^T \left[ I \quad \bar{P}^T \right] \nabla J^T, \tag{4.13a}$$

$$\delta u = B_0 x_a, \tag{4.13b}$$

where $x_a \in \mathbb{R}^{n_a}$ is the allocator state, $B_0 \in \mathbb{R}^{m \times n_a}$ is the matrix specifying with its columns the allocation directions in the input space, $\rho$ is a positive scalar which specifies the allocator convergence speed and $K$ is a symmetric positive definite matrix which can be used to distribute the allocation effort in the different directions. The allocat-
Figure 4.1: Block diagram of the *allocated closed loop*. The allocator (the dark dashed block, in which the block $A$ corresponds to (4.13a) and (4.13b)) takes only the controller output as an input. The cost function $J$ penalizes large values of $\delta y$, namely the difference between the actual output $y$ and the steady-state output value of the *original closed loop*.

The allocator block is connected to the closed loop system through

$$u = y_c + \delta u, \quad (4.14a)$$

$$u_c = y - \bar{P}\delta u, \quad (4.14b)$$

as shown in Fig. 4.1. We will refer to the interconnection (4.1), (4.2), (4.13), (4.14) as the *allocated closed loop system*. The additional output $\bar{P}\delta u$ is such that at the steady-state it corresponds to the output variation $\delta y$ caused by the allocator action. It is needed to make the allocation transparent to the given linear controller (4.2), ensuring that, at least at the steady-state, the controller input and output have the same values as in the *original closed loop*.

The following assumption on $J(\bar{u}, \delta \bar{y})$ and $\tilde{J}(\bar{x}_a)$ is made; in Sec. 4.3 we will see an example of a function satisfying the assumption and we’ll discuss about the properties which motivate its adoption.

**Assumption 4.2** The function $J(\bar{u}, \delta \bar{y})$ is continuously differentiable.
Moreover, for any fixed value of $\bar{y}_c$, the function $\tilde{J}(\bar{x}_a)$ is radially unbounded and strictly convex.

Let analyze first the behaviour of the allocator block (4.13) alone. The key property of the allocator (4.13) is that, for each constant input $\bar{y}_c$, it has a globally asymptotically stable equilibrium $\bar{x}_a$ coinciding with the global minimizer

$$x_a^* = \arg \min_{x_a \in \mathbb{R}^{n_a}} \tilde{J}(x_a).$$

(4.15)

From the definition of $\tilde{J}(x_a)$ we know that

$$\tilde{J}(x_a^*) = \min_{x_a \in \mathbb{R}^{n_a}} \tilde{J}(x_a)$$

(4.16)

$$= \min_{(u, \delta y) \in \mathcal{M}(\bar{y}_c)} J(u, \delta y)$$

(4.17)

where $\mathcal{M}(\bar{y}_c)$ is the manifold defined as

$$\mathcal{M}(\bar{y}_c) := \left\{ (u, \delta y) : \begin{bmatrix} u - \bar{y}_c \\ \delta y \end{bmatrix} \in \text{Im} \left( \begin{bmatrix} I \\ \bar{P} \end{bmatrix} B_0 \right) \right\}. \quad (4.18)$$

Once we have chosen a particular matrix $B_0$, the manifold $\mathcal{M}(\bar{y}_c)$ represents the set of all the possible couples $(\bar{u}, \delta \bar{y})$ we can obtain as steady-state by choosing different constant values for the free parameter $x_a$.

We formalize and prove this property in the following lemma.

**Lemma 4.1** Under Assumption 4.2, for any constant $\bar{y}_c$ the system (4.13) has a globally asymptotically stable (GAS) equilibrium $x = \bar{x}_a$ which is the minimizer of $J(\bar{u}, \delta \bar{y})$ constrained to the manifold $\mathcal{M}(\bar{y}_c)$.

**Proof.** Strict convexity of $\tilde{J}(x_a)$ with respect to $x_a$ implies that for any fixed $\bar{y}_c$ there exists a global minimum of $\tilde{J}(x_a)$; in view of the differentiability of $\tilde{J}(x_a)$, the corresponding global minimizer $x_a^*$ is the unique point where the gradient $\nabla \tilde{J}(x_a)$ is zero. It must be shown that the allocator has a unique equilibrium $\bar{x}_a$ coinciding with $x_a^*$. 


Since $x^*_a$ is a strict global minimum and $\tilde{J}(x_a)$ is radially unbounded, the function

$$V(\tilde{x}_a) := \tilde{J}(x_a) - \tilde{J}(x^*_a), \quad (4.19)$$

with $\tilde{x}_a = x_a - x^*_a$, is globally positive definite and radially unbounded; moreover, denoting by $\Delta > 0$ the minimum eigenvalue of $K > 0$, the time derivative

$$\dot{V} = -\rho \left( \nabla \tilde{J} \right) K \left( \nabla \tilde{J} \right)^T \leq -\rho \Delta |\nabla \tilde{J}|^2 \quad (4.20)$$

is globally negative definite since $\nabla \tilde{J} = 0$ if and only if $x_a = x^*_a$; hence, by standard results (see e.g. [23, Ch. 3]), $x^*_a$ is globally asymptotically stable.

Under additional assumptions on $\tilde{J}(x_a)$, the above stability property can be made stronger, as proved in the next lemma.

**Assumption 4.3** There exist positive constants $c$, $k_1$, $k_2$ and $k_3$ such that $V$ in (4.19) satisfies for (4.13a),

$$k_1 |\sigma x_a|^c \leq V(\sigma x_a) \leq k_2 |\sigma x_a|^c, \quad (4.21a)$$

$$\nabla V(\sigma x_a) \sigma x_a \leq -k_3 |\sigma x_a|^c. \quad (4.21b)$$

**Lemma 4.2** Under Assumptions 4.2 and 4.3, for any constant $\bar{y}_c$, the equilibrium $\bar{x}_a$ in Lemma 4.1 is globally exponentially stable (GES).

**Proof.** By standard results (see e.g. [23, Ch. 3]), condition (4.21) implies global exponential stability. □

Now we want to establish properties of the allocator (4.13) when suitably interconnected to the allocated closed loop. As in the case of weakly input redundant plants, general results can be found with a two time scales analysis, where overall stability properties of the whole interconnected system follow from asymptotic stability of the a priori
given closed loop (4.4) and of the allocator (4.13), provided that the allo-
cation speed $\rho$ is sufficiently small; however, since in the present case
the allocator (4.13) might be globally asymptotically stable but not
exponentially stable, a general result would require suitable growth
conditions to be satisfied (see [23, Ch. 9]). In order to simplify mat-
ters the following Theorem 4.1 will be directly geared towards the case
when the allocator (4.13) is globally exponentially stable.

**Theorem 4.1** Under Assumptions 4.1, 4.2 and 4.3, there exists $\bar{\rho} > 0$ such that for any $\rho \in (0, \bar{\rho})$, the allocated closed loop (4.1), (4.2),
(4.13), (4.14), is well posed and globally exponentially stable. More-
over, with constant exogenous signal $w$, its response converges to a
constant steady-state minimizing $J(\bar{u}, \delta \bar{y})$ constrained to the manifold $\mathcal{M}(\bar{y}_c)$.

**Proof.** The proof easily follows applying standard results on two time
scales systems (see e.g. [23, Ch. 9]) by considering the allocator dy-
namics (4.13a) as the slow subsystem, and the a priori given closed
loop (4.1), (4.2), (4.14), with (4.13b) as the fast subsystem. In par-
ticular, notice that for the fast subsystem global exponential stability
follows from linearity, whereas for the slow subsystem it follows from
Lemma 4.2 under Assumptions 4.2 and 4.3. \hfill \Box

**Remark 4.1** For the selection of matrix $B_0$ we must consider that
each of its columns corresponds to an “allocation direction”, which
will be dynamically weighted by a scalar component of $x_a$; at the
same time, if we choose $K$ as a diagonal positive definite matrix, we
can specify different speeds for each one of the directions given by $B_0$.

**Remark 4.2** In view of the fact that the relation $\delta \bar{y} = \bar{P} \delta \bar{u}$ is intrin-
sic to the plant (and not due to any specific allocation scheme) it is
worth to introduce the manifold $\mathcal{M}_0(\bar{y}_c) \supset \mathcal{M}(\bar{y}_c)$, defined as

$$
\mathcal{M}_0(\bar{y}_c) := \left\{ (u, \delta y) : \begin{bmatrix} u - \bar{y}_c \\ \delta y \end{bmatrix} \in \text{Im} \left( \begin{bmatrix} I \\ \bar{P} \end{bmatrix} \right) \right\},
$$

(4.22)
which corresponds to $\mathcal{M}(\tilde{y}_c)$ in (4.18) when $B_0 = I$, i.e. the allocator is given full authority over the plant input space $\mathbb{R}^m$ (instead of authority over a set of directions corresponding to $\text{Im}(B_0) \subset \mathbb{R}^m$). This is especially relevant for non input redundant plants (since for input redundant plants $B_0$ would generally be chosen as a matrix whose columns belong to $\ker \begin{bmatrix} B \\ D \end{bmatrix}$ or $\ker (\bar{P})$, and hence $\text{Im} (B_0) = \mathbb{R}^m$ would not hold) where $\bar{B}_0 = I$ is often used and so, under such a choice, the allocator yields the best possible steady-state under the obvious, necessary constraint that the input and output variations commanded by the allocator must be compatible with the steady-state characteristics of the plant expressed by $\bar{P}$.

**Remark 4.3** Though in general any input allocation will imply an output change in the case where $\text{rank}(\bar{P}) = m$ (with the entity of change on each scalar output determined by the trade-off induced by the cost function $J$), it is still possible to select $B_0$ in such a way to leave $\nu < m$ scalar outputs untouched at steady-state; this is very desirable in some applications, where some outputs can be considered “critical” and then should not move at all whereas larger tolerances are admissible on the remaining outputs. In order to achieve this, consider a selection matrix $S_y \in \mathbb{R}^{\nu \times p}$ obtained by selecting from a $p \times p$ identity matrix the $\nu$ rows corresponding to the outputs that must be left unchanged at steady-state; next, choose $B_0$ as a full column rank matrix such that

$$\text{Im}(B_0) = \ker (S_y \bar{P}).$$

(4.23)

In the same spirit, if it is desired to keep fixed also $\mu$ inputs, it is enough to consider an additional selection matrix $S_u \in \mathbb{R}^{\mu \times m}$ obtained by selecting from a $n_u \times m$ identity matrix the $\mu$ rows corresponding to the inputs that must be left unchanged; next, choose $B_0$
as a full column rank matrix such that

\[
\text{Im}(B_0) = \ker \begin{bmatrix} S_y \bar{P} \\ S_u \end{bmatrix}
\] (4.24)

(clearly, in order to do this it is assumed that \( \nu + \mu < m \), so that the indicated kernel has dimension larger than zero). Notice that in the last case, the allocator does not change the fixed inputs at any time (whereas only the steady state value of the outputs is unchanged in the previous case).

**Remark 4.4** The allocator described in Chap. 3 is actually a particular case of the one presented in this chapter. If the plant is actually *weakly redundant* we can select \( B_0 = B_\perp \) (see Chap. 3), obtaining \( \bar{P}B_0 = 0 \) so that the second argument in (4.12) vanishes. Then, if we select the cost function as \( J = u^TWu = (y_c + B_\perp w)^TW(y_c + B_\perp w) \), the new allocator equations (4.13), (4.14) give rise to the previous allocator equations (3.7), (3.3). So the new allocator generalizes the previous one to the case where there is no null space in \( \bar{P} \) and allows the choice of more general cost functions.

**Remark 4.5** The results presented in this chapter can be seen as extensions of those in Chap. 3. First of all, the considered cost function \( J(\bar{u}, \delta \bar{y}) \) doesn’t need to be quadratic, thus the designer has much more freedom in its choice, allowing to deal with more elaborated tasks. Second, the results are applicable to plants which are not even weakly input redundant; a key idea in order to make this second feature possible consists in determining the input allocation also on the basis of its effects on the steady-state output deviations \( \delta y \) from the steady-state induced by the a priori given controller (4.2). Third, the input to the plant is formally proven to converge to the constrained minimum of the function \( J(\bar{u}, \delta \bar{y}) \), possibly in an exponentially fast fashion.

With this approach, and by a judicious choice of \( J(\bar{u}, \delta \bar{y}) \) which suitably penalizes the \( \delta \bar{y} \) in addition to \( \bar{u} \), it is possible to trade-off
reduced deviations of the steady-state output induced by the unconstrained closed loop response meanwhile allowing for more favorable input values.

### 4.3 Piece-wise quadratic cost function

In this section we consider the problem of keeping the control inputs in their available range and suggest a possible selection of the cost function $J(\bar{u}, \delta y)$ which allows to satisfy a number of frequently occurring requirements for this aim.

A desirable behaviour for the allocation policy can be described as follows: we want the control system to have the same behaviour of the original closed loop (4.4) as long as the control inputs are far from their saturation levels. When the original closed loop would request (almost) unfeasible values for the control, we want the allocator to work for relaxing the control request in order to remain (or return as soon as possible) into the input feasible region.

To reach this goal we suggest, in particular, the following function (for brevity, the superscript $\bar{\cdot}$ is omitted)

$$J(u, \delta y) = \frac{1}{2} \sum_{i=1}^{m} w_{ui} \text{dz}(u_i)^2 + \frac{1}{2} \sum_{i=1}^{p} w_{yi} (\delta y_i)^2, \quad (4.25)$$

where $w_{ui} \geq 0$, $i = 1, \ldots, m$ and $w_{yi} > 0$, $i = 1, \ldots, p$. The deadzone function is defined for a vector $u \in \mathbb{R}^m$, as $\text{dz}(u) := \left[ \text{dz}(u_1) \ldots \text{dz}(u_m) \right]^T$, where $\text{dz}(u_i) = \text{sign}(u_i) \max\{0, |u_i| - 1\}$.

From the point of view of allocation and tuning, such a function has the following features:

- penalizes separately each $u_i$ and $\delta y_i$, according to the weight tuning parameters $w_{ui}$ and $w_{yi}$;
- it does not penalize $u_i$ as long as $u_i \in [-1, +1]$;
4.3 Piece-wise quadratic cost function

Figure 4.2: Each input is weighted with a piece-wise quadratic function. As long as an input remains inside his (safety) range $[-1, 1]$ it does not cost at all. Each output variation is weighted with a quadratic function. As long as all the inputs remain inside their (safety) range $[-1, 1]$, the only goal of the allocator is to reach as far as possible the behaviour of the original closed loop.

- penalizes $\delta y_i$ quadratically (hence large values of $\delta y_i$ are penalized much more than small values);

- penalizes $u_i$ quadratically when $u_i \notin [-1, +1]$.

The above points imply that priority is given to keep the $\delta y_i$’s small as long as the $u_i$’s satisfy $u_i \in [-1, +1]$; meanwhile, when the $u_i$’s are sufficiently outside the interval $[-1, +1]$, if the $w_{ui}$’s are sufficiently big with respect to the $w_{yi}$’s, then priority is given to keep the $u_i$’s close to the interval $[-1, +1]$, even at the price of larger $\delta y_i$’s.

Another important property of the cost function (4.25) concerns the convergence. If the cost function in (4.25) is used, in fact, then exponential convergence can be easily established, as stated in the following Theorem 4.2. To prove it we need first to introduce the following lemma.
For compactness, define

\[ W_u := \text{diag}(w_{u1} \ldots w_{um}), \quad (4.26a) \]
\[ W_y := \text{diag}(w_{y1} \ldots w_{yp}), \quad (4.26b) \]
\[ \bar{W}_y := \sqrt{W_y}, \quad (4.26c) \]
\[ M := B_0^T \bar{P}^T W_y \bar{P} B_0, \quad (4.26d) \]

so that (4.25) can be rewritten as

\[ J(u, \delta y) = \frac{1}{2} \text{dz}(u)^T W_u \text{dz}(u) + \frac{1}{2} \delta y^T W_y \delta y. \quad (4.27) \]

In the choice of the parameters, we make the following assumption

**Assumption 4.4** The matrix \( \bar{P}B_0 \) is full column rank.

**Lemma 4.3** If Assumption 4.4 is satisfied then the cost function in (4.25) satisfies Assumptions 4.2 and 4.3.

**Proof.** First of all we notice that (4.27) is continuously differentiable. Then we recall that \( \bar{J}(x_a) = J(\bar{y}_c + B_0 x_a, \bar{P} B_0 x_a) \) and so, by substituting in (4.27) we obtain

\[
\bar{J}(x_a) = \frac{1}{2} \text{dz}(\bar{y}_c + B_0 x_a)^T W_u \text{dz}(\bar{y}_c + B_0 x_a) + \frac{1}{2} x_a^T M x_a
\geq \frac{1}{2} x_a^T M x_a. \quad (4.28)
\]

Since \( \bar{W}_y \bar{P} B_0 \) is full column rank by hypothesis, then \( M \) is clearly positive definite and so \( \bar{J}(x_a) \) results to be radially unbounded. As for \( \bar{J}(x_a) \) being a strictly convex function, notice that it is the sum of two terms: \( \frac{1}{2} \sum_{i=1}^{p} b_i \delta y_i^2 \), with \( \delta y = \bar{P} B_0 x_a \), which is strictly convex since it is twice differentiable with positive definite Hessian matrix given by \( M \); the second term \( \frac{1}{2} \sum_{i=1}^{m} a_i \text{dz}(u_i)^2 \), with \( u = y_c + B_0 x_a \) is convex because it is the composition of the function \( \sum_{i=1}^{m} a_i \text{dz}(u_i)^2 \), which is convex in \( u \), with the affine function \( u = y_c + B_0 x_a \). As for \( \frac{1}{2} \sum_{i=1}^{m} a_i \text{dz}(u_i)^2 \) being convex in \( u \), notice that it is convex being a linear combination with positive coefficient of the convex functions \( \text{dz}(u_i)^2 \). So Assumption 4.2 has been proved.
As for (4.21a), recalling the definition of $V$ in (4.19) and noting that due to the expression (4.25) the function $V$ is continuously differentiable, piecewise quadratic and positive definite, it is easy to see that

$$x_a^T M x_a \leq V(w) \leq x_a^T (M + B_0^T W u B_0) x_a,$$

(4.29)

and then (4.21a) holds with $c = 2$, $k_1 = \lambda$ and $k_2 = \lambda > \lambda$, where $\lambda$ is the minimum eigenvalue of $M$ and $\lambda$ is the maximum eigenvalue of $M + B_0^T W u B_0$.

As for (4.21b), since (4.25) (and hence $V$) is piecewise quadratic and its gradient is piecewise linear and globally Lipschitz, then the function $\dot{V} = -\rho \left( \nabla \tilde{J} \right) K \left( \nabla \tilde{J} \right)^T$ must also be piecewise quadratic. By the Lipschitz property of $\nabla \tilde{J}$, the Hessian of $V$ exists almost everywhere; moreover, it is a piecewise constant (matrix) function of $x_a$ taking on only a finite number of values (due to the form of (4.25), and, in particular, due to the fact that the “piecewise” structure depends on the deadzone functions, which implies that at most $3^m$ different regions exist); finally, each of these finite values is a positive definite matrix (since strict convexity and piecewise quadraticity of $V$ imply that the Hessian is positive definite wherever it exists). Choosing $k_3 > 0$ as the minimum among all the eigenvalues of the Hessian of $V$, it is easy to see that the bound (4.21b) must hold.

Theorem 4.2 Under Assumptions 4.1 and 4.4, if $J(u, \delta y)$ is taken as in (4.25) then there exists $\bar{\rho} > 0$ such that for any $\rho \in (0, \bar{\rho})$, the input allocated closed loop (4.1), (4.2), (4.14), (4.13) with constant exogenous input $w = [r^T d^T]^T$ is globally exponentially stable (GES), and its response converges to a constant steady-state minimizing $J(\bar{u}, \delta \bar{y})$ constrained to the manifold $M(\bar{y}_c)$.

Proof. By 4.3, Assumption 4.4 implies that Assumptions 4.2 and 4.3 are satisfied; in turn, satisfaction of Assumptions 4.1, 4.2 and 4.3 allows to deduce the result from Theorem 4.1.

Remark 4.6 The result in Theorem 4.2 still holds if the desired “penalty free” interval for the $u_i$ has the more general form $[u_i, \bar{u}_i]$.
and if it is desired to put different penalty weights on positive and negative deviations $\delta y_i$, as well as if it is desired to weight differently the values of $u_i$ below $\underline{u}_i$ or above $\bar{u}_i$. In fact, in order to achieve this, it is enough to replace (4.25) by

$$J(u, \delta y) = \sum_{i=1}^{m} a_i \zeta_i(u_i)^2 + \sum_{i=1}^{p} b_i \xi_i(\delta y_i)^2,$$  \hfill (4.30)

where the terms $\zeta_i(u_i)$ are given by

$$\zeta_i(u_i) := \begin{cases} 
\bar{a}_i (u_i - \bar{u}_i), & \text{if } u_i > \bar{u}_i, \\
0, & \text{if } u_i \in [\underline{u}_i, \bar{u}_i], \\
\underline{a}_i (u_i - \underline{u}_i), & \text{if } u_i < \underline{u}_i,
\end{cases}$$  \hfill (4.31)

with $0 \leq \underline{a}_i \leq \bar{a}_i$, and the terms $\xi_i(\delta y_i)$ are given by

$$\xi_i(\delta y_i) := \begin{cases} 
\bar{b}_i \delta y_i, & \text{if } \delta y_i \geq 0, \\
\underline{b}_i \delta y_i, & \text{if } \delta y_i < 0,
\end{cases}$$  \hfill (4.32)

with $0 \leq \underline{b}_i \leq \bar{b}_i$. 

Figure 4.3: The single terms of the weighting function (4.30) allow to take into account asymmetric safety ranges for the inputs, and to penalize differently the inputs out of range on different sides. The terms related to output variations are still quadratic, but with different coefficients for positive or negative deviations.
Remark 4.7 Still a further extension is possible if it is desired to allocate the input while keeping also the consequent output values \( \bar{y}_i \) (and not only the consequent variations \( \delta \bar{y}_i \) of the output values). In this case, the additional term \( \sum_{i=1}^{p} c_i \psi_i(y_i)^2 \) can be inserted in (4.30), where the functions \( \psi_i(\cdot) \) have an expression similar to the functions \( \zeta_i(\cdot) \) in (4.31), but the limits \( u_i, \bar{u}_i \) on \( u \) are replaced by suitable limits \( y_i, \bar{y}_i \) on \( y \), and the constants \( a_i, \bar{a}_i \) are replaced by possibly different constants \( c_i, \bar{c}_i \).

4.3.1 Simulations

In this section we show some simulation results to better understand the behaviour of the allocator designed in this chapter.

Example 4.1 We consider a plant like the one in (4.1), with \( n = 4, m = 2, p = 3, n_d = 0 \) and matrices

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix} = \begin{bmatrix}
0.31 & 0.13 & 0.45 & 0.45 & -0.08 & 0.15 \\
0.40 & -0.41 & 0.46 & -0.02 & 0.41 & -0.47 \\
-0.38 & -0.23 & -0.35 & 0.30 & 0.29 & 0.34 \\
0.41 & 0.04 & 0.47 & -0.36 & 0.45 & 0.43 \\
0.17 & -0.11 & 0.20 & -0.46 & 0 & 0 \\
0.25 & 0.15 & -0.47 & -0.41 & 0 & 0 \\
0.24 & -0.33 & -0.23 & 0.32 & 0 & 0
\end{bmatrix}.
\] (4.33)

The plant is clearly under-actuated, in fact we have \( m < p \). The available range of the actuators is \([-1, 1]\). The following control block ensures the set-point regulation for every reference \( r \in \text{Im}(\bar{P}) \). The
controller is like (4.2) with matrices

\[
\begin{bmatrix}
Ac & Br \\
Cc & Dr
\end{bmatrix} =
\begin{bmatrix}
-2.16 & 1.11 & 3.85 & 0.89 & 0.09 & 0.14 & -0.93 & -1.15 & 0 & 0 & 0 \\
-0.48 & -0.37 & 2.16 & 0.19 & -0.20 & -0.59 & -3.00 & 0.66 & 0 & 0 & 0 \\
-0.28 & -0.10 & -0.77 & 0.25 & 0.44 & -0.08 & 0.59 & -0.56 & 0 & 0 & 0 \\
-1.13 & 0.70 & 2.52 & -0.09 & 0.60 & -0.18 & -4.25 & -0.88 & 0 & 0 & 0 \\
-3.40 & 0.13 & -2.38 & -2.30 & -2.57 & 0.17 & -0.78 & -0.62 & 0 & 0 & 0 \\
0.40 & 0.54 & 0.01 & 0.25 & 0.17 & -1.21 & 0.62 & -0.78 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.09 & 0.76 & 0.64 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1.79 & -0.36 & -1.75 & -1.43 & -1.42 & 1.15 & 0 & 0 & 0 & 0 & 0 \\
-2.48 & 0.04 & -2.07 & -1.88 & -2.14 & -0.46 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix},
\]

\(B_c = -B_r\) and \(D_c = -D_r\). Our goal is to keep the input signal \(|u|_\infty < 1\) (at least at the steady-state). An allocator like (4.13) is designed choosing the piece-wise quadratic cost function (4.28) with parameters \(a_i = b_j = 0.5\), \(\overline{a}_i = \overline{b}_j = 1\), \(\overline{u}_i = -\overline{u}_i = 0.7\) (\(\forall i = 1 \ldots m\), \(\forall j = 1 \ldots p\)), \(\rho = 2\) and \(K = I\), where \(I\) represents the identity matrix. Notice that the safe region \([\underline{u}_i, \overline{u}_i]\) is smaller than the available range, in order to allow the allocator to turn on. A first

![Figure 4.4: With a small reference the control input \(u\) is identical with and without the allocator.](image-url)
4.3 Piece-wise quadratic cost function

Figure 4.5: With a small reference $r$ the tracking error $e$ is identical with and without the allocator. In both cases it doesn’t go to 0 because the reference $r \notin \text{Im}(\bar{P})$.

Figure 4.6: With a small reference $r$ the output deviation $\delta y$ is identically zero.

Figure 4.7: With a small reference $r$ the cost function value remains constant.
Dynamic input allocation for non input-redundant systems

simulation shows how the closed loop responds to a step reference of amplitude $r = [0.1 \ 0.1 \ 0.1]^T$, with and without the allocator. With this reference the control effort required remains in the safe region (Fig. 4.1), so the allocator does’t react at all and all the signals, with and without the allocator, are identical. Notice that the tracking error $e$ (Fig. 4.1) doesn’t go to 0 because the reference $r \notin \text{Im}(\bar{P})$.

In a second simulation we choose $r = [0.3 \ 0.3 \ 0.3]^T$. In this case

Figure 4.8: With a medium reference $r$ the control input $u$ without allocator exceeds the available range. With the allocator it is maintained lower.

Figure 4.9: With a medium reference $r$ the tracking error $e$ with the allocator is a little bigger than without just during the transient, while asymptotically it converges to the same value.

the original closed loop requests a control effort $u$ (Fig. 4.1) which exceeds the available range, but just in the transient. So the allocator
4.3 Piece-wise quadratic cost function

Figure 4.10: With a medium reference $r$ the output deviation $\delta_y$ is different from 0 just during the transient, while asymptotically it goes to 0.

Figure 4.11: With a medium reference $r$ the allocator maintains the cost function at a value that is almost everywhere lower than without the allocator. Asymptotically the cost goes to 0.

at the beginning reacts by trying to keep the control small, but then slowly turns off because its action is no more needed and the allocated input returns asymptotically to the original trajectory. This is evident also in Fig. 4.1, where we can see the output variation $\delta y$ due to the allocator action increases when the input is over the bounds, but then returns to 0; looking at the cost function value in Fig. 4.1, we see that the allocator action actually succeeds to maintain smaller the value of $J$. A last simulation shows a more critical situation, in which the reference $r = [0.90.90.9]^T$ requests to the original closed loop a control
Figure 4.12: With a large reference $r$ the control input $u$ without allocator exceeds the available range and remains there even at the steady-state. With the allocator, instead, after a transient, it is brought back inside the range.

Figure 4.13: With a large reference $r$ the tracking error $e$ with the allocator is a little bigger than without. Anyways we can see that at the steady state this difference is acceptable.

effort which remains over the limits also at the steady-state. In this case the allocator reacts by relaxing the request in order to bring back the control input inside the available region Fig. 4.1. Differently from the previous situation, now the output variation $\delta y$ remains different from 0 (Fig. 4.1) also at the steady-state, allowing to maintain the cost function (Fig. 4.1) smaller with respect to the original control system.
4.4 Trading input variations for tracking error variations

In this section we propose a different scheme for the input allocation, more suitable within a set-point regulation setting, and prove its convergence properties. We also present simulation results to assess the steady-state and transient performances obtained with the novel scheme, as compared to the previous approach.

Consider the plant (4.1) controlled by an a-priori given linear con-
Dynamic input allocation for non input-redundant systems

\[ \dot{x}_c = A_c x_c + B_c e, \quad (4.34a) \]
\[ y_c = C_c x_c + D_c e, \quad (4.34b) \]

connected through

\[ u = y_c, \quad (4.35a) \]
\[ e = r - y, \quad (4.35b) \]

where \( e \) is the tracking error.

The input allocation scheme proposed in Sec. 4.2 (see Fig. 4.16 aims

![Figure 4.16: Block diagram of the allocated closed loop in the case of set-point regulation. It corresponds to choosing \( B_c = -B_r \) and \( D_c = -D_r \) in (4.2).](image)

at keeping the value of the plant inputs inside a desirable region, without modifying too much the steady-state output response of the system under the action of (4.2). A different point of view is considered in this section, by redefining the problem as keeping the value of the plant inputs inside a desirable region, meanwhile ensuring a small tracking error. To this aim, the modified allocator structure in Fig. 4.17 is considered, and the second argument \( \delta \bar{y} \) in the considered cost function \( J \) is replaced by \( \bar{e} \). Notice that the allocator equations still have the form (4.13), although in this case the gradient of \( J \) will depend
4.4 Trading input variations for tracking error variations

Figure 4.17: Block diagram of the error driven allocated closed loop.

The allocator (the dark block) takes both the controller input and output signals as an input. The cost function $J$ penalizes large values of $e$, namely the difference between the actual output $y$ and the reference value $r$.

on $u$ and $e$ (instead of $u$ and $\delta y$). The connection to the closed loop is described by

$$u = y_c + \delta u, \quad (4.36a)$$
$$e = y - r. \quad (4.36b)$$

At the steady-state the relations $\bar{u} = \bar{y}_c + B_0 \bar{x}_a$ and $\bar{e} = \bar{P}B_0 \bar{x}_a + \bar{P}\bar{y}_c - r$ hold. So, if we make explicit the dependence of $J(\bar{u}, \bar{e})$ on $\bar{x}_a$ and we consider $r$ constant and $\bar{y}_c$ as a fixed parameter (so that dependence on $r$ and $\bar{y}_c$ will not be indicated), it is possible to define the function

$$\tilde{J}_e(\bar{x}_a) := J(\bar{y}_c + B_0 \bar{x}_a, \bar{P}B_0 \bar{x}_a + \bar{P}\bar{y}_c - r). \quad (4.37)$$

The following theorem formally states some relevant properties of the proposed allocator scheme; in order to distinguish between the allocator scheme in Fig. 4.16 and the one in Fig. 4.17, the former will be called output driven allocator whereas the latter will be called error driven allocator.

**Assumption 4.5** The closed loop system (4.1), (4.34), (4.35) is well posed and exponentially stable.
Dynamic input allocation for non input-redundant systems

**Assumption 4.6** The function $J(\bar{u}, \bar{e})$ is continuously differentiable. Moreover, for any fixed value of $\bar{y}_c$, the function $\tilde{J}_e(\bar{x}_a)$ is radially unbounded and strictly convex.

If assumption 4.6 is satisfied we know that the function $\tilde{J}_e(\bar{x}_a)$ has exactly one minimizing point $x_a^*$ and we can define the function

$$V(x_a) := \tilde{J}_e(x_a) - \min_s \tilde{J}_e(s). \quad (4.38)$$

We will refer to the system given by the interconnection of (4.1), (4.34), (4.36) and (4.13) as the *error driven input allocated closed-loop system*. Now we can state the main properties of this new configuration, formalizing them in the next theorem.

**Theorem 4.3** Under Assumptions 4.5, 4.6 and 4.3 (applied to the function (4.38)) there exists $\bar{\rho} > 0$ such that for any $\rho \in (0, \bar{\rho})$ and for any constant exogenous signals $r$ and $d$, the error driven input allocated closed-loop system is globally exponentially stable, and its response converges to a constant steady-state minimizing $J$ constrained to the manifold $\mathcal{M}(\bar{y}_c)$ in (4.18); moreover the steady-state response induced by the error driven allocator coincides with the response induced by the output driven allocator for any constant exogenous signals $r$, $d$, if and only if the controller (4.2) ensures steady-state error-free tracking (i.e. $\bar{y} = r$).

**Proof.** The proof is identical to that of theorem (4.2) once we notice that $\delta \bar{e} = \bar{y} + \delta \bar{y} - r = \delta \bar{y} + (\bar{y} - r)$ is just a translation of $\delta \bar{y}$ by a constant value. \qed

**Remark 4.8** The Theorem 4.3 basically shows that the error driven allocator has the same properties of the output driven allocator. Anyways the last point of the theorem shows that, while there is no (steady-state) advantage in using the error driven allocator instead of the output driven allocator when the controller (4.2) already ensures steady-state error-free tracking, the corresponding steady-state
responses will be different when this is not the case (e.g. in all the cases in which the plant is under-actuated and \( r \notin \text{Im}(\bar{P}) \)); hence, the use of \( J(u, e) \) instead of \( J(u, \delta y) \) will in general lead to better results whenever it is desired to have small values of \( \bar{e} \) and not simply small deviations \( \delta \bar{y} \) from the output \( \bar{y} \) induced by the controller (in fact, the small deviation \( \delta \bar{y} \) traded off with the variation \( \delta \bar{u} \) by the output driven allocator might be in the direction of increasing \( \bar{e} \)).
Dynamic input allocation for non input-redundant systems
Chapter 5

Output regulation of input-redundant systems

5.1 Introduction

We present in this chapter preliminary results and research directions concerning the output regulation problem for over-actuated linear systems.

In Chap. 3 we addressed the problem of exploiting the input redundancy of an over-actuated plant, in order to optimize the control input in some sense, and we saw that a possible way is that of inserting an allocator block in the closed-loop. The results presented in Chap. 3 are valid as long as the exogenous signals, namely the reference and the disturbances, are constant in time or, at least, converging to a constant value. In the case of sinusoidal references or disturbances, for example, those results are no longer valid. In order to overcome this obstacle we try in this chapter to handle the problem from an output regulation perspective.
In the output regulation problem the goal is to find a control law such that the plant asymptotically tracks a given reference signal, despite the presence of disturbances, when both the reference and the disturbances are generated by a known exosystem. In the standard regulation theory two cases are distinguished: the full information problem and the output feedback problem. In the former both the plant and the exosystem states are supposed to be available for control, while in the latter just the plant output is available. In this chapter the full information case is considered.

When the plant is input redundant, the solution to the problem is not unique and infinitely many steady-state trajectories exist which ensure the asymptotic tracking. For this reason it makes sense to look for a strategy which allows to choose the “best one”, in some sense.

In the proposed solution, the intrinsic redundancy in the plant model is exploited by parameterizing all solutions of the ensuing regulator equations and performing a static or dynamic optimization on the space of solutions. This approach effectively shapes the non-unique steady-state of the system so that the long-term behavior optimizes a given performance index. In particular, nonlinear cost functions that account for constraints on the inputs are considered. Examples are given to illustrate and validate the proposed methodology.

The results presented in this chapter appeared in [24]. Basic ideas on which they rely already appeared in the previous works [15] and [16].

\subsection{5.2 The full information regulator problem for over-actuated systems}

Consider the following linear plant:

\begin{equation}
\mathcal{P}:
\begin{align*}
\dot{x} &= Ax + Bu + Pw \\
e &= Cx + Du + Qw
\end{align*}
\end{equation}
5.2 The full information regulator problem for over-actuated systems

with state $x \in \mathbb{R}^n$, control input $u \in \mathbb{R}^m$ and performance output $e \in \mathbb{R}^p$. The signal $w \in \mathbb{R}^q$ is assumed to be generated by the following exosystem

$$S : \dot{w} = Sw.$$  \hspace{1cm} (5.2)

Following standard regulation theory [25, 26], $(A, B, C, D)$ is refereed to as the realization of the plant model and $(S, P, Q)$ as the realization of the exosystem. The following assumptions define the class of plant and exosystem models considered in this chapter:

**Assumption 5.1**

1. The plant model is over-actuated, that is, $m > p$;
2. The matrices $B$ and $C$ satisfy rank $B = m$ and rank $C = p$;
3. The pair $(A, B)$ is stabilizable;
4. The matrix $S$ is semi-simple (that is, it has only simple eigenvalues) and spec $S \subset \mathbb{C}^0$.

Item 2 of Assumption 5.1 is made to avoid trivialities and overlap with previous results. The case in which $B$ is rank-deficient, which corresponds to having the strong input redundancy (see Chap. 3) for $(A, B)$, can be handled separately from the weak redundancy exploited here.

### 5.2.1 Problem Statement and Preliminaries.

The problem addressed in this chapter is the design of a full information (possibly nonlinear) regulator that is capable of exploiting the redundancy stated at item 1 of Assumption 5.1 to induce a desirable selection of the control input $u$ (in a sense to be specified).

As a possible selection of a function to be optimized, we consider in Sec. 5.4 function whose minimization corresponds to keeping the steady-state input far from the saturation limits.
As pointed out in the previous chapters, the use of input allocation should be seen as synergistic with anti-windup techniques, since the latter account for saturation during transients, whereas the former addresses steady-state saturations.

This must be done while guaranteeing internal stability of the closed-loop system when the exosystem is disconnected and the asymptotic tracking requirement \( \lim_{t \to \infty} e(t) = 0 \) when the exosystem is active.

As customary, by full information it means that both \( x \) and \( w \) are available for measurement. Here, it is also assumed that \( \mathcal{P} \) and \( \mathcal{S} \) are known exactly.

A standard sufficient condition for the solvability of the regulator problem (which becomes necessary if mild assumptions on parametric uncertainties affecting the plant matrices are considered) is given by the following:

**Assumption 5.2** [geometric version]

1. The plant model is right-invertible, i.e., the output \( e \) is functionally controllable from the input \( u \);

2. The set of transmission zeros of \( (A, B, C, D) \) is disjoint from the spectrum of \( S \).

An algebraic version of Assumption 5.2, known as Davison condition, can also be formulated. Let \( P_\Sigma(s) \) denote the *system matrix* of system (5.1), that is,

\[
P_\Sigma(s) := \begin{bmatrix}
    A - sI & B \\
    C & D
\end{bmatrix}.
\]

(5.3)

Recall that system (5.1) is left invertible if and only if \( \text{rank} P_\Sigma(s) = n + m \) (as a polynomial matrix), and it is right invertible if and only if \( \text{rank} P_\Sigma(s) = n + p \). Obviously, system (5.1) is not left-invertible.

The values of \( \bar{s} \in \mathbb{C} \) for which the complex valued matrix \( P_\Sigma(\bar{s}) \) has rank less than the rank of \( P_\Sigma(s) \) as a polynomial matrix constitute the
system zeros, which include all the transmission zeros plus a subset of the input decoupling zeros (eigenvalues of the unreachable subsystem) and the output decoupling zeros (eigenvalues of the unobservable subsystem). The algebraic version of Assumption 5.2 is then stated as follows:

**Assumption 5.2** [algebraic version] rank \( P_\Sigma(\lambda) = n + p \), for all \( \lambda \in \text{spec} \ S \).

Finally, we recall a few geometric concepts that will be used in the sequel. By \( V^* \subset \mathbb{R}^n \), we denote the weakly unobservable subspace for \( P \), that is, the set of initial conditions for which there exists an input function such that the ensuing output function is identically zero. It is well known (see [19]) that \( V^* \) is the largest subspace \( V \subset \mathbb{R}^n \) such that

\[
\begin{bmatrix}
A \\
C
\end{bmatrix} V \subset (V \times 0) + \text{im} \begin{bmatrix}
B \\
D
\end{bmatrix},
\]

or equivalently the largest subspace \( V \subset \mathbb{R}^n \) such that there exists \( F \in \mathbb{R}^{m \times n} \) ensuring

\[
(A + BF)V \subset V, \quad (C + DF)V = 0.
\]

A matrix \( F \) satisfying (5.5) is called a friend of \( V \). Similarly, we denote by \( R^* \subset \mathbb{R}^n \) the controllable weakly unobservable subspace\(^1\) of \( P \), that is, the set of initial conditions for which there exists an input function able to steer the state to zero in finite time while keeping the output function identically zero. Obviously, \( R^* \subset V^* \); moreover, any friend of \( V^* \) is also a friend of \( R^* \) [19, Th. 7.14].

5.3 Regulator architecture and properties

It is well known (see, for instance, [27, Ch. 1]) that the structure of a full-information regulator comprises:

\(^1\)When \( D = 0 \), \( V^* \) and \( R^* \) are usually termed respectively the largest controlled-invariant subspace and the largest controllability subspace contained in \( \ker C \) (see [26]).
1. a steady-state control action $u_{ss}(w)$ capable of inducing an identically zero output $e$ along a suitable steady-state trajectory $x_{ss}(w)$ of the plant, and

2. a stabilizing control action $\tilde{u}$ in feedback from the mismatch $\tilde{x} = x - x_{ss}(w)$, capable of stabilizing the steady-state trajectory at the previous item.

In the over-actuated case, since the plant model fails to have a unique inverse (recall that $P$ is necessarily not left-invertible), redundancy can be exploited in the generation of the steady-state pair $(x_{ss}(w), u_{ss}(w))$. For linear models, this corresponds to selecting appropriately

$$x_{ss}(w) = \Pi w, \quad u_{ss}(w) = \Gamma w$$

(5.6)

among the infinitely many solutions $(\Pi, \Gamma)$ of the regulator (or Francis) equations:

$$
\begin{align*}
\Pi S &= A\Pi + B\Gamma + P \\
0 &= C\Pi + D\Gamma + Q.
\end{align*}
$$

(5.7)

According to the Proposition 5.1, all steady-state pairs in (5.6) can be generated by exploiting a basis of the space of all solutions of the homogeneous Francis equation

$$
\begin{align*}
\Pi S &= A\Pi + B\Gamma \\
0 &= C\Pi + D\Gamma.
\end{align*}
$$

(5.8)

**Proposition 5.1** Under Assumption 5.1, all solutions to the Francis equations (5.7) are parametrized as

$$
\Pi(\theta) = \Pi_p + \sum_{i=1}^{s} \theta_i \Pi_i, \quad \Gamma(\theta) = \Gamma_p + \sum_{i=1}^{s} \theta_i \Gamma_i
$$

(5.9)

by the parameter vector $\theta = [\theta_1 \cdots \theta_s]^T \in \mathbb{R}^s$, where $s = (m - p)q$, $X_p = [\Pi_p^T \Gamma_p^T]^T$ is any solution of (5.7) whereas $X_i = [\Pi_i^T \Gamma_i^T]^T$, $i = 1, \ldots, s$ are linearly independent matrices spanning the space of solutions of (5.8).
5.3 Regulator architecture and properties

The next result is key to the selection of the stabilizing component of the regulator:

Proposition 5.2 Each solution \( X_i = [\Pi_i^T \Gamma_i^T]^T \) of (5.8) satisfies \( \text{im} \Pi_i \subset \mathcal{R}^* \), \( i = 1, \ldots, s \).

Proof. See Sec. 5.6.

□ The results of Propositions 5.1 and 5.2, as well as the general structure of a (static) full-information regulator mentioned above, suggest the architecture of the regulator with dynamic allocation shown in Figure 5.1. In particular, dynamic allocation of the control input \( u \) is performed by acting on the allocation parameter \( \theta \). Ideally, this must be accomplished without affecting the tracking performance – that is, preserving the asymptotic properties of the error signal \( e(\cdot) \) as well as the controlled-invariance of the subspace \( x = \Pi(\theta)w \).

Once the steady-state trajectory ensuring \( e(t) \equiv 0 \) is computed as in (5.6), (5.9), it is then possible to design the feedback stabilizer, which according to the scheme in Figure 5.1, provides the feedback

![Figure 5.1: The over-actuated regulator control scheme with dynamic input allocator.](image-url)
signal $\tilde{u}$ within the selection

$$u = u_{ss}(w, \theta) + \tilde{u}.$$  \hfill (5.10)

In particular, substituting (5.10), (5.6), (5.9) in the plant dynamics (5.1), exploiting (5.7), and defining $\tilde{x} = x - x_{ss}(w, \theta)$, the following dynamic equations are derived for the error system:

$$\begin{align*}
\dot{\tilde{x}} &= A\tilde{x} + B\tilde{u} - \sum_{i=1}^{s} \hat{\theta}_i \Pi_i w \\
e &= C\tilde{x} + D\tilde{u}.
\end{align*}$$  \hfill (5.11)

Given (5.11), it is quite natural to select the input $\tilde{u}$ as the linear feedback

$$\tilde{u} = K\tilde{x} = K(x - x_{ss}(w, \theta)), $$ \hfill (5.12)

where $K$ is designed in one of the following two ways:

1. as any stabilizing gain for the pair $(A, B)$, under the condition that $\dot{\theta}(t) \equiv 0$, that is the allocation parameter $\theta$ is statically optimized and is kept constant;

2. as a stabilizing gain for $(A, B)$ with the property that the transfer matrix from $\xi = [\xi_1, \cdots, \xi_s]$ (with $\xi_i \in \mathbb{R}^q$) to $e$ for the system

$$\begin{align*}
\dot{x} &= A\bar{x} + B\bar{u} + \sum_{i=1}^{s} \Pi_i \xi_i \\
e &= C\bar{x} + D\bar{u}.
\end{align*}$$  \hfill (5.13)

is identically zero; this second choice will allow for nonstationary selections of the allocation parameter $\theta(t)$ without affecting the tracking performance.

The two possible selections above and the desirable properties of the ensuing “Over-Actuated Full Information Regulator” in Figure 5.1 are formally stated in Theorem 5.1.

**Theorem 5.1** Consider the plant (5.1) and the exosystem (5.2) satisfying Assumptions 5.1 and 5.2. Moreover, consider the controller
5.4 Selection of the allocation policy

In the previous section we have introduced an architecture for the Over-Actuated Full Information Regulator of Figure 5.1 and two design choices for the stabilizer $K$ without commenting on the selection of the allocator block $A$. In particular, via Theorem 5.1, we provided two design techniques for synthesizing the matrices of the regulator parametrically in $\theta$ in such a way that the tracking performance is preserved with constant selections of $\theta(t) = \bar{\theta}$ (item 1 of Theorem 5.1) and with differentiable selections of $\theta(t)$ (item 2 of Theorem 5.1). In this section, we will propose a few techniques to select the allocation parameter $\theta$ (namely the allocator block $A$ in Figure 5.1), in such a way to induce desirable properties of the closed-loop signals.

It should be recognized that the techniques proposed here are quite
straightforward and intuitive constructions, while more involved designs for $\theta$ might be achieved by possibly developing more in the direction highlighted in the next remark.

**Remark 5.1** Consider the (parametric in $\theta$) control law (5.10), (5.12), (5.6), (5.9) proposed in the previous section. Taking into account the matrix selection in (5.9), this control law can be rewritten as the following affine function of $\theta$:

$$u = \Gamma(\theta)w + K(x - \Pi(\theta)w) = Kx + (\Gamma_p - K\Pi_p)w + \Psi(w)\theta,$$

where

$$\Psi(w) = \begin{bmatrix} (\Gamma_1 - K\Pi_1)w & \cdots & (\Gamma_s - K\Pi_s)w \end{bmatrix}.$$  

(5.14a)

Based on the representation (5.14) of the control input $u$, it is possible to design static or dynamic selections of $\theta$ aiming at keeping the control $u$ as small as possible, possibly based on its saturation limits or based on other plant input performance specifications. In this chapter, two different selections of $\theta$ will be considered. In the first one a constant value of $\theta$ is pre-computed off-line by optimizing a suitable performance index. In the second one, an (almost) piecewise constant $\theta(t)$ is obtained by an on-line optimization algorithm which adapts $\theta(t)$ with the goal of reaching the minimum of the same performance index. This give raise to a time-varying selection of $\theta(t)$ which converges in finite time to a constant value. More general time-varying selections of $\theta$ and dynamic selections along similar lines to those developed in [14, 18] will be investigated in future work.

For simplicity, the following assumption is introduced. Its satisfaction can be easily verified by checking if the ratios among (the imaginary parts of) the eigenvalues of $S$ are all rational numbers.

**Assumption 5.3** The exosystem (5.2) generates periodic responses, that is, there exists $T > 0$ such that for any $w(0)$ and for all $t \geq 0$, $w(t + T) = w(t)$. 

5.4 Selection of the allocation policy

5.4.1 An off-line selection of a constant $\theta$

The first strategy for optimizing the input allocation whenever Assumption 5.3 holds arises from recognizing that for each initial condition $w_0$ of the exosystem (5.2) (and for each value of $\theta$), a unique periodic steady-state control input $u_{ss}(t) = \Gamma(\theta)w$ is defined whenever $\theta$ is kept constant. Indeed, due to Assumption 5.3, the response $w(t)$ is periodic and only depends on the initial condition.

A sensible problem then corresponds to the one of selecting $\theta$ as the minimizer of the following cost function:

$$J(\theta, w(t)) = \max_{t \in [0, T]} \left( 1 - \frac{u_{ss,i}(w(t))}{\bar{u}_i} \right) \left( \frac{u_{ss,i}(w(t))}{\bar{u}_i} - 1 \right),$$

which satisfies $J(\theta, w(t)) = J(\theta, w(0))$ for all $\theta$ and for all $t \geq 0$, because $w(t) = w(t + T)$ and because the function corresponds to the maximum over the whole period.

The rationale behind the cost (5.15) is to maximize the worst case distance of any input from its saturation level. This is actually carried out by normalizing the saturation levels so that the percentage of the available input range at all inputs is maximized.

Note that the cost function takes into account the possible periodic nature of the steady-state input by computing the worst case distance over the whole period of the steady-state plant input.

Based on the cost function (5.15), the approach proposed here is to optimize the selection of $\theta$ based on the measurement of the initial value of $w$ (or, equivalently, its value at any time $t \geq 0$). In particular, assuming that it is possible to solve offline the following optimization problem:

$$\theta^*(w_0) = \arg\min_{\theta \in \mathbb{R}^s} J(\theta, w_0),$$

a constant value of $\theta(t) = \theta^*(w_0)$ can be adopted and, according to item 1 of Theorem 5.1, for any $K$ stabilizing the pair $(A, B)$, the
asymptotic tracking performance is retained and the worst distance of the worst input from its saturation value is maximized. This fact is illustrated in Example 5.1 in the next section.

5.4.2 An on-line iterative selection of $\theta$

The constant selection of $\theta$ proposed above suffers from two main issues: 1) it might be nontrivial, in general, to explicitly compute the maximizer of the cost function (5.15); 2) even in cases where this maximizer could be computed, this should be parametrized with respect to $w$, and this might be expensive in terms of storage of a suitable grid of optimal values for $\theta$.

To address the above problems, we propose here a gradient based strategy which allows to optimize online the value of $\theta(t)$. The functional dependence of $\theta$ on time is restricted to be a smooth but almost piecewise constant signal. In particular, we fix a small scalar $\delta \ll T$ denoting the time used to smoothly transition between two subsequent periods (namely, time intervals of length $T$) where $\theta(t) = \theta_{[k]}$ is constant. Then we define:

$$
\theta(t) := \theta_{[k]}, \quad \forall t \in \mathcal{T}_k, \ k \in \mathbb{N},
$$

$$
\mathcal{T}_k := [(T + \delta)k, (T + \delta)(k + 1)].
$$

Consider the index

$$
J_{[k]} := J(\theta, w(t), k) = \max_{i \in \{1, \ldots, m\}} J_i(\theta, w(t)),
$$

$$
J_i(\theta, w(t)) = \left(1 - \frac{u_{ss,i}(w(t))}{\bar{u}_i}\right)\left(\frac{u_{ss,i}(w(t))}{\bar{u}_i} - 1\right)
$$

which corresponds to (5.15) but evaluated during the $k-$th period.

In the $\delta$-small interval $[(T + \delta)k + T, (T + \delta)(k + 1)]$, between the end of the $k$-the period and the beginning of the $(k + 1)$-the period, $\theta(t)$ is smoothly transferred from $\theta_{[k]}$ to:

$$
\theta_{[k+1]} = \theta_{[k]} - \alpha_{[k]}d_{[k]},
$$
where, letting \( i_{[k]}^* \in \{1, \ldots, m\} \) denote the index of the input component such that the maximum in (5.18) is achieved and letting \( t_{[k]} \) denote the corresponding time, the update direction \( d_{[k]} \) and the update step length \( \alpha_{[k]} \) are chosen as

\[
\begin{align*}
    d_{[k]} &= \frac{\nabla_\theta J_{i_{[k]}^*}(\theta_{[k]}, w(t_{[k]}))}{\left\| \nabla_\theta J_{i_{[k]}^*}(\theta_{[k]}, w(t_{[k]})) \right\|}, \\
    \beta_{[k]} &= \tau (J_{[k]} + 1)^2 \left\| \nabla_\theta J_{i_{[k]}^*}(\theta_{[k]}, w(t_{[k]})) \right\|, \\
    \alpha_{[k]} &= \begin{cases} 
        \beta_{[k]}, & \beta_{[k]} \geq \alpha, \\
        0, & \beta_{[k]} < \alpha
    \end{cases}
\end{align*}
\]

with \( \tau, \alpha \) being positive constants. The factor \((J_{[k]} + 1)^2\) is useful to get smoother convergence, since it gets closer to zero (thus reducing the step length and inducing more cautious updates) when the components \( u_{ss,i} \) get closer to zero, that is far from the bounds on the input. The parameter \( \alpha \) is used to stop the updates when the update step becomes sufficiently small; in this way, after a finite number of iterations the value of \( \theta \) remains constant and the result in Theorem 5.1 applies, yielding asymptotic tracking with a reduced excursion of the steady-state input. Example 5.2 in Sec. 5.5 shows the effectiveness of the proposed recipe.

### 5.5 Simulation examples

In this section two numerical examples are presented in order to show the two different strategies proposed. In both cases the simulations are made without the saturation limits actually activated. The purpose of the simulations is, in fact, just to show that in certain situations the input allocation allows to have feasible steady-state control inputs, which would be unfeasible otherwise. If the saturation constraints were actually present, an anti-windup action could be used to manage the unfeasible control required by the controller and the allocator.
during the transient. Anyway in this work for the sake of simplicity the saturation constraint are not actually imposed, but just taken into account as soft constraints to evaluate the eventual steady-state feasibility of the control signals.

A first numerical example is presented to illustrate the item 1 of Theorem 5.1.

**Example 5.1** Consider the plant and exosystem described by the matrices:

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix} = \begin{bmatrix}
-0.02 & 0.41 & 0.35 & -0.76 & -0.31 \\
-0.10 & 0.50 & 0.31 & -0.00 & 0.17 \\
0.29 & -0.44 & -0.67 & 0.91 & -0.55 \\
0.50 & -0.48 & 0.01 & 0 & 0
\end{bmatrix}
\]

\[S = \begin{bmatrix}
0 & 15.78 \\
-15.78 & 0
\end{bmatrix}\]

\[
\begin{bmatrix}
P \\
Q
\end{bmatrix} = \begin{bmatrix}
0.78 & -0.72 \\
0.91 & -0.70 \\
0.09 & -0.48 \\
0.68 & -0.49
\end{bmatrix}
\]

All the matrices are generated randomly, except for the matrix \(S\), which is characterized by two imaginary eigenvalues \(\lambda_1 = \lambda_2^* = j\frac{2\pi}{T}\), with \(T\) generated randomly and given by \(T = 0.39\). Moreover, select \(\bar{u} = [\begin{array}{c}
-30 \\
-20
\end{array}]^T\) and \(\overline{u} = [30\;30]^T\) in (5.18). Assumptions 5.1 and 5.2 are both satisfied, so the results of Theorem 5.1 can be applied. The dimension of the free parameter \(\theta\) is given by \(s = (m - p)q = (2 - 1)2 = 2\). Solving the Francis equation, the particular and homogeneous
solutions can be found to be

\[
\begin{bmatrix}
\Pi_p \\
\Gamma_p
\end{bmatrix} =
\begin{bmatrix}
-1.47 & 0.91 \\
-0.08 & -0.09 \\
1.72 & -1.11 \\
20.70 & 29.03 \\
0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
\Pi_1 \\
\Pi_2 \\
\Gamma_1 \\
\Gamma_2
\end{bmatrix} =
\begin{bmatrix}
-0.43 & -0.93 & 0.93 & -0.43 \\
-0.37 & -0.82 & 0.82 & -0.37 \\
2.95 & 5.39 & -5.39 & 2.95 \\
-47.82 & 28.98 & -28.98 & -47.82 \\
70.93 & -42.44 & 42.44 & 70.93
\end{bmatrix} \times 10^{-2}.
\]

A first stabilizing matrix

\[ K_{NF} = \begin{bmatrix} -68.49 & -380.02 & -80.01 \\ 0.06 & -65.40 & -2.68 \end{bmatrix} \]

is computed using the Matlab command \texttt{place} with the goal of assigning the eigenvalues of the closed-loop matrix \((A + BK_{NF})\) at \([-8, -10, -12]\). Then, Theorem 5.1 guarantees that using \(K = K_{NF}\), for any constant value of \(\theta\), asymptotic tracking is ensured. In particular, according to the construction given in Sec. 5.4.1, we select a constant value of \(\theta\) optimizing the performance index \(J(\theta)\) in (5.15) when initializing the exosystem (5.2) from the initial condition \(w_0 = [10]^T\).

Figure 5.2 shows a numerical evaluation of the optimal value of \(\theta\), based on the level sets of \(J(\theta)\).

Note that the minimizer is \(\theta^*(w_0) = [618]^T\). This proves that the redundancy in the regulator problem can be effectively used to obtain an improved usage of the steady-state plant inputs. This is even more evident in Figure 5.3, where the steady-state input trajectory corresponding to \(w_0\) is plotted on the inputs space for the two cases of \(\theta = 0\) (blue circles) and \(\theta = \theta^*\) (red dots). In the former case, with \(\theta = 0\), the input cannot satisfy the saturation constraint (represented by the green dashed lines), while in the latter case, with \(\theta = \theta^*\), the input trajectory lies entirely in the available range and maximizes the worst case distance from the saturation limits.
Figure 5.2: Example 5.1. The cost $J$ in (5.15) as a function of the parameter $\theta \in \mathbb{R}^2$. The minimizer is $\theta^* = [020]^T$ (red dot), different from zero (blue circle).

A second stabilizing matrix

$$K_F = \begin{bmatrix} -217.50 & -240.26 & -84.16 \\ 337.82 & 339.42 & 124.90 \end{bmatrix}$$

is computed to assign the same set of eigenvalues assigned by $K_{NF}$ while being a friend of $R^*$.

With $K = K_F$, the subspace spanned by the matrix

$$R = \begin{bmatrix} -0.69 & -0.00 \\ -0.71 & 0.02 \\ 0.01 & 0.99 \end{bmatrix}$$

is invariant.

In Figure 5.4 two different simulations, respectively with $K = K_{NF}$ (black dash-dotted) and $K = K_F$ (blue solid), are shown. In both of them the exosystem starts from the same initial condition.
5.5 Simulation examples

Figure 5.3: Example 5.1. The values of $u_{ss}$ in the case $\theta = 0$ (blue circles), and for the optimal case $\theta = \theta^*$ (red dots). In the former case the steady-state trajectory is infeasible for the input constraints while in the latter case it is feasible.

$$w_0 = [10]^T$$ used in the previous steps, but the plant starts from an initial condition $x_0 = x_{ss}(w_0, \theta) + \tilde{x}_0$, with $\tilde{x}_0 = C^T = [0.50 - 0.480.01]^T$, which does not belong to the invariant subspace of the steady-state trajectories. In both simulations $\theta = \theta^*$ is used, so that the steady-state trajectories of both the input components are within their saturation limits (green dashed horizontal lines). Since $\theta$ is constant, item 1 of Theorem 5.1 applies and so both with the friend matrix $K_F$ and with the non friend one $K_{NF}$, the inputs converge, after a transient, to the steady-state trajectory $u_{ss}(w)$ (red dashed curves) and the error $e$ converges to 0 (lower trace in the figure).

Example 5.2 Consider the plant and exosystem described by the ma-
Figure 5.4: Response using \( \theta = \theta^* \) with \( K = K_F \) (blue solid) and with \( K = K_{NF} \) (black dash-dotted). In both cases the error \( e \) converges to 0 (lower plot) and \( u \) converges to \( u_{ss}(w) \) (red dashed), because \( \theta \) is constant.

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix} =
\begin{bmatrix}
-0.01 & 0.80 & 0.56 & -0.19 & 0.88 \\
-0.02 & -0.26 & -0.22 & -0.80 & 0.91 \\
-0.32 & -0.77 & -0.51 & -0.73 & 0.15 \\
-0.88 & -0.53 & -0.29 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & -7.65 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 3.82 \\
0 & 0 & 0 & -3.82 & 0 \\
0.56 & -0.83 & 0.85 & 0.55 & -0.02
\end{bmatrix}
\]

\[
S =
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 7.65 & 0 & 0 \\
0 & 0 & 0 & 0 & 3.82 \\
0 & 0 & 0 & -3.82 & 0 \\
-0.96 & 0.29 & -0.09 & 0.48 & -0.63 \\
-0.91 & 0.46 & 0.09 & -0.62 & -0.26 \\
-0.66 & 0.29 & -0.40 & 0.37 & 0.25
\end{bmatrix}
\]
Figure 5.5: Example 5.2. Inputs $u$ and $u_{ss}$ (top two plots), error $e$ (middle trace), cost function $J$ in (5.18) (middle-bottom trace) and parameter $\theta$ (bottom trace).

All the matrices are generated randomly, except for the matrix $S$, which is characterized by an eigenvalue in the origin $\lambda_1 = 0$ and two pairs of imaginary eigenvalues $\lambda_2 = \lambda_3^* = j\frac{2\pi}{T_1}$ and $\lambda_4 = \lambda_5^* = j\frac{2\pi}{T_2}$ with $T_2 = 2T_1$ and $T_1 = 0.82$ generated randomly. Moreover, select $\overline{u} = [-30\ -20]^T$ and $\overline{u} = [30\ 30]^T$ in (5.18). Assumptions 5.1 and 5.2 are both satisfied, so the results of Theorem 5.1 can be applied. The dimension of the free parameter $\theta$ is given by $s = (m - p)q = (2 - 1)5 = 5$. Solving the Francis equation, the particular and homogeneous
solutions can be found to be

\[
\begin{bmatrix}
\Pi_p \\
\Gamma_p
\end{bmatrix} = \begin{bmatrix}
-0.26 & -0.52 & 0.65 & 0.35 & -0.18 \\
1.49 & -0.58 & 0.54 & 0.39 & 0.23 \\
0 & -0.23 & -0.03 & 0.10 & 0.03 \\
-2.52 & 0 & 0 & 0 & 0 \\
-0.81 & -5.34 & -4.86 & -0.17 & 2.00
\end{bmatrix} 10^{-2}
\]

\[
\begin{bmatrix}
\Pi_1 \\
\Gamma_1
\end{bmatrix} = \begin{bmatrix}
3.65 & 0.09 & -3.65 & -0.13 & 0.79 \\
-20.83 & -0.24 & 2.43 & 0.16 & -0.51 \\
26.66 & 0.14 & 6.56 & 0.10 & -1.44 \\
2.18 & 78.35 & -8.60 & -8.84 & 0.85 \\
2.49 & 48.83 & -7.46 & -5.58 & 0.99
\end{bmatrix} 10^{-2}
\]

\[
\begin{bmatrix}
\Pi_2 \\
\Gamma_2
\end{bmatrix} = \begin{bmatrix}
-7.49 & 2.34 & -0.81 & -1.48 & 1.97 \\
42.70 & -1.60 & 0.42 & 1.18 & -1.17 \\
-54.67 & -4.13 & 1.66 & 2.30 & -3.80 \\
-4.48 & 24.20 & 47.65 & -25.38 & -9.99 \\
-5.12 & 16.42 & 29.20 & -16.65 & -5.07
\end{bmatrix} 10^{-2}
\]

\[
\begin{bmatrix}
\Pi_3 \\
\Gamma_3
\end{bmatrix} = \begin{bmatrix}
5.60 & 3.10 & 0.72 & -0.00 & -1.93 \\
-31.92 & -2.03 & -0.63 & -0.18 & 1.29 \\
40.86 & -5.62 & -1.02 & 0.34 & 3.46 \\
3.35 & -6.34 & 68.26 & 20.60 & -5.69 \\
3.82 & -2.16 & 42.90 & 12.81 & -4.65
\end{bmatrix} 10^{-2}
\]

\[
\begin{bmatrix}
\Pi_4 \\
\Gamma_4
\end{bmatrix} = \begin{bmatrix}
-2.24 & 0.41 & -0.44 & 7.15 & -1.37 \\
12.77 & -0.30 & 0.27 & -4.91 & 0.22 \\
-16.35 & -0.70 & 0.83 & -12.55 & 3.71 \\
-1.34 & 10.68 & 7.61 & 35.50 & 72.01 \\
-1.53 & 6.89 & 4.48 & 26.20 & 44.01
\end{bmatrix} 10^{-2}
\]

\[
\begin{bmatrix}
\Pi_5 \\
\Gamma_5
\end{bmatrix} = \begin{bmatrix}
2.88 & 0.27 & 0.77 & 1.88 & 6.83 \\
-16.44 & -0.14 & -0.53 & -0.59 & -4.75 \\
21.04 & -0.55 & -1.36 & -4.56 & -11.89 \\
1.72 & -15.77 & 8.08 & -67.15 & 39.98 \\
1.97 & -9.66 & 5.48 & -40.69 & 28.81
\end{bmatrix} 10^{-2}
\]

Two different stabilizing matrices,

\[
K_{NF} = \begin{bmatrix}
224.99 & -259.22 & 251.22 \\
105.78 & -134.41 & 125.14
\end{bmatrix}
\]

\[
K_F = \begin{bmatrix}
160.13 & -412.32 & 275.49 \\
94.41 & -260.23 & 169.56
\end{bmatrix}
\]
are computed: the former using the Matlab command `place` with the goal of assigning the eigenvalues of the closed-loop matrix $(A+BK_{NF})$ at \{-8, -10, -12\}; the latter assigning the same eigenvalues while being a friend of $R^*$. With $K = K_F$, the subspace spanned by the matrix

$$R = \begin{bmatrix} -0.56 & -0.00 \\ 0.72 & 0.48 \\ 0.39 & -0.87 \end{bmatrix}$$

is invariant.

Then, Theorem 5.1 guarantees that using $K = K_{NF}$, for any constant value of $\theta$, asymptotic tracking is ensured. In particular, we select $\theta$ according to the construction given in Sec. 5.4.2. Figure 5.5 shows a numerical simulation of the closed-loop system, with initial conditions $x_0 = [0.630.580.28]^T$, $w_0 = [3.6302.12212]^T$ and $\theta_0 = 0$, repeated in both the cases with $K = K_{NF}$ (blue dashed-dotted trace) and $K = K_F$ (black solid trace).

In order to show how the allocation law (5.19) responds to disturbances, at time $t = 50$ the first component of the exosystem state $w_1$ jumps to the value $w_1 = -6.4$. Both the initial condition $w_0$ and the value of $w$ after the jump, define infeasible ($J > 0$) steady-state trajectories $u_{ss}$ (red solid bold trace) for the input $u$. The allocator block changes the free parameter $\theta$ and makes the actual cost index $J(\theta)$ decrease to negative values. Indeed $u_{ss}$ moves back into the feasible input range. With both the choices for the stabilizer $K$, the actual input $u$ reaches the steady-state trajectory $u_{ss}$ after a transient due to both the initial condition $\tilde{x}_0 \neq 0$ and to the adaptation of $\theta$, as described by (5.13). The advantage in using the friend matrix and relying on item 2 of Theorem 5.1 appears from the middle plot of the figure, which shows that the friend matrix does not cause the impulsive transients experienced on the error $e$ when $\theta$ varies according to the adaptation law.
5.6 Proof of the main results

Proof of Proposition 5.1: The Francis equation (5.7) can be cast as the Hautus equation [28]
\[ A_1 X - A_2 X S = E, \]  
(5.20)
where
\[ E = \begin{bmatrix} P \\ Q \end{bmatrix}, \quad A_1 = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad A_2 = \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}. \]
The left-hand side of equation (5.20) defines a linear operator \( \mathcal{H} : \mathbb{R}^{(n+m)\times q} \to \mathbb{R}^{(n+p)\times q}. \) The set of all solutions of the linear equation \( \mathcal{H}(X) = E \) above is given by the linear variety \( \mathcal{L} = X_p + \ker \mathcal{H}, \) where \( X_p = [\Pi_p^T \Gamma_p^T]^T \) is a particular solution. It has been shown in [28] (see also [27]) that the operator \( \mathcal{H} \) is surjective if and only if
\[ \text{rank} \left( A_1 - A_2 \lambda \right) = n + p, \quad \forall \lambda \in \text{spec } S, \]
which is indeed implied by Assumption 5.2. Consequently, \( \dim \ker \mathcal{H} = (m - p)q = s, \) consistently with the fact that the solution of (5.7) is unique if \( m = p. \) It follows that all solutions of the Francis equation (5.7) are given by (5.9). \( \square \)

Proof of Proposition 5.2: A few intermediate results (of independent interest, hence cast as lemmas) are needed to build the proof. We begin by noticing the following result, which is a direct consequence of the first two items in Assumption 5.1.

Lemma 5.1 Let \( \rho := \dim \mathcal{R}^*. \) Then, \( \rho \neq 0. \)
Proof of Lemma 5.1 Since \( m > p, \) system (5.1) is not left invertible. Item 2 of Assumption 5.1 implies that \( \text{rank} [B^T D^T]^T = m, \) thus the matrix \( [B^T D^T]^T \) corresponds to an injective linear operator. By [19, Th. 8.26], it follows that \( \mathcal{R}^* = \{0\}. \) \( \square \)
Let \( \nu \geq \rho \) denote the dimension of \( \mathcal{V}^* \) and choose matrices \( T, F \) and \( G \) as follows:
• $T \in \mathbb{R}^{n \times n}$ is an invertible matrix such that its first $\rho$ columns span $\mathcal{R}^*$ and its first $\nu$ columns span $\mathcal{V}^*$;

• $G = \begin{bmatrix} G_1 & G_2 & G_3 \end{bmatrix} \in \mathbb{R}^{m \times m}$ is an invertible matrix such that $\text{im} G_1 = B^{-1} \mathcal{R}^* \cap \ker D$ and $\text{im} \begin{bmatrix} G_1 & G_2 \end{bmatrix} = \ker D$;

• $M = \begin{bmatrix} M_1 & M_2 \end{bmatrix} \in \mathbb{R}^{p \times p}$ be an invertible matrix such that $M_1DG_3 = 0$, $M_2DG_3 = I$;

• $F \in \mathbb{R}^{m \times n}$, $F = F_0 + F^*_\mathcal{V}$, with $F^*_\mathcal{V}$ being a friend of $\mathcal{V}^*$ and $F_0 = -G_3M_2(C + DF^*_\mathcal{V})$.

Apply the regular feedback transformation $u = Fx + \bar{G} \hat{u}$, the output coordinate change $\bar{e} = Me$ and the coordinate change $z = T^{-1}x$ to system (5.1) to obtain

\begin{align*}
\dot{z} &= \bar{A}_Fz + \bar{B} \bar{u} + \bar{P}w \\
\bar{e} &= \bar{C}_Fz + \bar{D} \bar{u} + \bar{Q}w
\end{align*}

(5.21)

where $\bar{A}_F = T^{-1}(A + BF)T$, $\bar{B} = T^{-1}BG$, $\bar{P} = T^{-1}P$, $\bar{C}_F = M(C + DF)T$, and $\bar{D} = MDG$ and $\bar{Q} = MQ$, having the structure

\[
\begin{bmatrix}
\bar{A}_F & \bar{B} & \bar{P} \\
\bar{C}_F & D & \bar{Q}
\end{bmatrix}
= \begin{bmatrix}
\bar{A}_{11} & \bar{A}_{12} & \bar{A}_{13} & 0 & \bar{B}_{11} & \bar{B}_{12} & \bar{B}_{13} & \bar{P}_1 \\
0 & \bar{A}_{22} & \bar{A}_{23} & 0 & \bar{B}_{22} & \bar{B}_{23} & \bar{P}_2 \\
0 & 0 & \bar{A}_{33} & 0 & \bar{B}_{32} & \bar{B}_{33} & \bar{P}_3 \\
0 & 0 & 0 & 0 & 0 & 0 & I & \bar{Q}_1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{Q}_2
\end{bmatrix}
\]

(5.22)

where, in particular, $\bar{A}_{11} \in \mathbb{R}^{\rho \times \rho}$, $\bar{A}_{22} \in \mathbb{R}^{(\nu - \rho) \times (\nu - \rho)}$, $\bar{A}_{33} \in \mathbb{R}^{(n - \nu) \times (n - \nu)}$, and $\bar{B}_{11} \in \mathbb{R}^{\rho \times p_1}$. Recall that, by definition, the pair $(\bar{A}_{11}, \bar{B}_{11})$ is controllable, hence the spectrum of $\bar{A}_{11}$ is assignable by a friend of $\mathcal{R}^*$ (hence, a friend of $\mathcal{V}^*$) [19, Th. 4.18]. Conversely, spec $\bar{A}_{22}$ is the set of all eigenvalues of $\bar{A}_F^* \mathcal{V}^*$, that can not be assigned by a friend of $\mathcal{V}^*$, which coincides with the set of transmission zeros of $(A, B, C, D)$. 


The matrix $\tilde{A}_{22}$ represents, in the given coordinates, the map induced on $\mathcal{V}^*/\mathcal{R}^*$ by $A$ [19, Th. 7.14]. It should also be noted that the pair

$$
\begin{bmatrix}
\tilde{A}_{22} & \tilde{A}_{23} \\
0 & \tilde{A}_{33}
\end{bmatrix},
\begin{bmatrix}
\tilde{B}_{22} \\
\tilde{B}_{32}
\end{bmatrix}
$$

is stabilizable, as it can be easily verified by the PBH test.

**Lemma 5.2** Under Assumption 5.2, the subsystem described by the quadruple $(\tilde{A}_{33}, \tilde{B}_{32}, \tilde{C}_{13}, 0)$ above is square and has no finite invariant zero (hence no finite transmission zero).

**Proof of Lemma 5.2** The system matrix $\tilde{P}_2(s)$ of (5.21) is related to the original $P_2(s)$ of (5.1) by pre- and post-multiplication by constant invertible matrices as follows:

$$
\begin{bmatrix}
\tilde{A} - sI & \tilde{B} \\
\tilde{C} & \tilde{D}
\end{bmatrix} = \begin{bmatrix}
T^{-1}AT + T^{-1}BFT - sI & T^{-1}BG \\
MCT + MDFT & MDG
\end{bmatrix}
$$

$$
= \begin{bmatrix}
T^{-1} & 0 \\
0 & M
\end{bmatrix} \begin{bmatrix}
A - sI & B \\
C & D
\end{bmatrix} \begin{bmatrix}
T & 0 \\
FT & G
\end{bmatrix}
$$

hence rank $\tilde{P}_2(s) = \text{rank } P_2(s)$ as polynomial matrices.

Considering the structure of the matrices $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ and rearranging the columns of $\tilde{P}_2(s)$, the following matrix is obtained

$$
\begin{bmatrix}
\tilde{A}_{11} - sI & \tilde{B}_{11} & \tilde{A}_{12} & \tilde{A}_{31} & \tilde{B}_{12} & \tilde{B}_{13} \\
0 & 0 & \tilde{A}_{22} - sI & \tilde{A}_{32} & \tilde{B}_{22} & \tilde{B}_{23} \\
0 & 0 & 0 & \tilde{A}_{33} - sI & \tilde{B}_{32} & \tilde{B}_{33} \\
0 & 0 & 0 & \tilde{C}_{13} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I
\end{bmatrix}
$$

which by Assumption 5.2 (and the fact that only invertible transformations were used) must still have rank $n+p$ as a polynomial matrix. This imposes that the block

$$
\begin{bmatrix}
\tilde{A}_{33} - sI & \tilde{B}_{32} \\
\tilde{C}_{13} & 0
\end{bmatrix}
$$

$$
\begin{bmatrix}
\tilde{B}_{33} \\
0
\end{bmatrix}
$$

$$
\begin{bmatrix}
I
\end{bmatrix}
$$
is full row rank, so that $\bar{B}_{32}$ must have at least $p - \rho_3$ independent columns, where $\rho_3$ is the number of columns of $\bar{B}_{13}$. The fact that $\bar{B}_{32}$ has exactly $p - \rho_3$ columns is shown as follows: Suppose that $\bar{B}_{32}$ contains more than $p - \rho_3$ columns. For any $s = s_0 \in \mathbb{R}$ there would exist $x_{30}, \bar{u}_{20}$ such that

$$
\begin{bmatrix}
\bar{A}_{33} - s_0 I & \bar{B}_{32} \\
\bar{C}_{13} & 0
\end{bmatrix}
\begin{bmatrix}
x_{30} \\
\bar{u}_{20}
\end{bmatrix} = 0,
$$

so that the state $x_a := \begin{bmatrix} 0 & 0 & x_{30}^T \end{bmatrix}^T$, $x_a \notin \mathcal{V}^*$, and the input $u_a := \begin{bmatrix} 0 & u_{20}^T & 0 \end{bmatrix}^T$ satisfy

$$
\begin{bmatrix}
A \\
C
\end{bmatrix} x_a + \begin{bmatrix}
B \\
D
\end{bmatrix} u_a = s_0 \begin{bmatrix}
x_a \\
0
\end{bmatrix},
$$

implying that the subspace $\text{span}(x_a) + \mathcal{V}^*$ would be strictly larger than $\mathcal{V}^*$ and satisfy (5.4); but this contradicts the maximality of $\mathcal{V}^*$. A similar reasoning applies for $s = s_0 \in \mathbb{C} \setminus \mathbb{R}$ by considering the subspace generated by the real and the imaginary part of $x_a$. \qed

We are finally in the position to complete the proof of Proposition 5.2. With reference to the proof of Proposition 5.1, consider $\ker \mathcal{H}$, the space of all solutions of equation (5.8). It is easy to see that $\ker \mathcal{H}$ is invariant with respect to regular feedback transformations (and, obviously, coordinate transformations). In particular, $X$ solves (5.8) if and only if $\bar{X} := (\bar{\Pi}; \bar{\Gamma}) = (T^{-1}\Pi; G^{-1}\Gamma - G^{-1}F\Pi)$ solves

$$
\bar{\Pi} S = \bar{A} F \bar{\Pi} + \bar{B} \bar{\Gamma} \\
0 = \bar{C} F \bar{\Pi} + \bar{D} \bar{\Gamma}
$$

which is the homogeneous Francis equation associated to the transformed system (5.21). Consequently, it is enough to prove the proposition for the system (5.21); this amounts to showing that $\bar{\Pi}_2 = 0$.
and $\Pi_3 = 0$, where $\Pi = [\Pi_1^T \, \Pi_2^T \, \Pi_3^T]^T$ and $\Gamma = [\Gamma_1^T \, \Gamma_2^T \, \Gamma_3^T]^T$ are partitioned according to the given decomposition of system (5.21). The homogeneous Francis equation (5.24) in the new coordinates reads as a set of three equations

$$\begin{align*}
\Pi_1 S &= A_{11} \Pi_1 + A_{12} \Pi_2 + A_{13} \Pi_3 + B_{11} \Gamma_1 + B_{12} \Gamma_2 \\
&\quad + B_{13} \Gamma_3 \\
\Pi_2 S &= A_{22} \Pi_2 + A_{23} \Pi_3 + B_{22} \Gamma_2 + B_{23} \Gamma_3 \\
\Pi_3 S &= A_{33} \Pi_3 + B_{32} \Gamma_2 + B_{33} \Gamma_3 \\
0 &= C_{13} \Pi_3 \\
0 &= \Gamma_3
\end{align*}$$

(5.25)

(5.26)

(5.27)

The equality $\Gamma_3 = 0$ is exactly (5.27).

Since by Lemma 5.2 the quadruple $(\tilde{A}_{33}, \tilde{B}_{32}, \tilde{C}_{13}, 0)$ has no transmission zeros and the corresponding system is square, the solution $(\Pi_3; \Gamma_2)$ of the Francis equation (5.26) is unique, and by inspection it is given by $(\Pi_3; \Gamma_2) = (0; 0)$. Hence, equation (5.25) reduces to the homogeneous Sylvester equation

$$\Pi_2 S = A_{22} \Pi_2$$

in the unknown $\Pi_2$. Since by virtue of Assumption 5.2 (in particular, item 2 for the geometric version) the spectra of $S$ and $A_{22}$ are disjoint, the unique solution of this Sylvester equation is $\Pi_2 = 0$.

\textbf{Proof of Theorem 5.1} The boundedness of trajectories is proven first. In both cases, the dynamics of (5.1) under (5.10), (5.12) and (5.6), can be written as $\dot{x} = (A + BK)x + f(t, w)$, with $\dot{x} = (A + BK)x$ being linear time invariant and exponentially stable, so that the response in $x, e$ will be bounded (in $L_\infty$) as long as the driving term $f(t, w)$ is bounded (in $L_\infty$), which is true by item 4 of Assumption 5.1 and the boundedness hypothesis of $\theta(\cdot)$ in item 2 of Theorem 5.1.

The properties of $e(\cdot)$ are now established. In case 1, the system is time-invariant, and $x_{ss}(w, \theta), u_{ss}(w, \theta)$ are a solution of the system
corresponding to $e(t) = 0$ for all $t$; this proves that for all initial condition satisfying $x(0) = \Pi(\theta(0))w(0)$ the error satisfies $e(t) \equiv 0$. Moreover, by linearity and the fact that $(A + BK)$ is Hurwitz, the error between any motion of the closed-loop system and the motion with $x(0) = \Pi(\theta(0))w(0)$ must converge to zero, and then also $e(\cdot) = 0$ must converge to zero.

In case 2, the same reasoning adopted for case 1) shows that the motion obtained keeping $\theta(t) = \theta(0)$ (and then $\dot{\theta}(t) = 0$) for all $t \geq 0$ would guarantee $e(t) = 0$ for all $t \geq 0$. By (5.11), the effect of having $\dot{\theta}(t) \neq 0$ is to have an additional input on $\tilde{x}$ whose range is inside the range of the matrices $\Pi_i$, which in turn by Proposition 5.2 is contained in $\mathcal{R}^*$; so the choice of $K$ as a friend of $\mathcal{R}^*$ guarantees that $\tilde{x}$ remains in $\mathcal{R}^*$, so that also $e(t) = 0$ for all $t \geq 0$. □
Part II

Applications to magnetic confinement of plasma in tokamaks
Chapter 6

Introduction Part II

In this part we apply some of the dynamic allocation techniques presented in the previous part to deal with two different control problems in the field of nuclear fusion.

Nuclear fusion is the process by which the nuclei of two light atoms such as hydrogen are fused together to form a heavier (helium) nucleus, with energy produced as a by-product. The primary challenge of fusion is to confine a gas comprised of ionized hydrogen isotopes, called a plasma, while it is heated and its pressure increases to initiate and sustain fusion reactions. There are different ways to confine the plasma. In magnetic confinement machines the property of magnetic fields of exerting a force on the plasma is exploited.

Tokamaks (see [29]) are magnetic confinement devices with toroidal shape (Fig. 6.1). Magnetic fields are produced by currents in large coils and used to confine the plasma within a toroidal vacuum vessel. Several of these magnetic coils serve additional purposes of shaping, heating, and driving current in the plasma. In order to have nuclear reactions, in fact, the plasma must be carried to high temperatures (of the order of 100 millions C). In particular the toroidal field (TF) coils
confine the plasma inside the chamber, the central solenoid is used to
drive the plasma current and heat the plasma (ohmic heating) and the
poloidal field (PF) coils are used to control plasma position and shape.

To describe plasma position and shape, the last closed magnetic
flux line inside the chamber is considered as the plasma boundary.
In old tokamaks plasmas with circular shape cross section were stud-
ied, while in modern ones the plasma column has usually a vertically
elongated shape (see Fig 6.2). In the first case the plasma is said to
be in “limiter” configuration, because the plasma boundary actually
touches the first wall in a point called limiter point. In the second case
the magnetic flux is characterized by a saddle topology and the plasma
boundary corresponds to the flux line passing by the saddle point (or
X-point), which is called the separatrix. In this configuration, called
“divertor” configuration, the plasma does not touch directly the first wall. Anyways the separatrix intersects the chamber wall, in a region called divertor, on two points called “strike points”.

To describe plasma position and shape in a quantitative way, various geometric descriptors are used in different tokamaks. Sometimes just a few global parameters are used, like the plasma centroid position, the minor and major radius and the “elongation”, which is the ratio between the vertical and the horizontal axis, i.e. a measure of how much squeezed is the plasma. Sometimes more detailed parameters sets are used, like the “gaps”, namely distances between the plasma boundary and the first wall measured along some given lines.

The need for achieving better performance in present and future tokamak devices is pushing plasma control (see [30]) to gain increasing
importance in tokamak engineering. High performance in tokamaks is achieved by plasmas with elongated poloidal cross-section. A strong motivation to improve plasma control is that, in order to obtain the best performance out of a device, it is always necessary to maximize the plasma volume within the available space; hence, the ability to control the shape of the plasma while ensuring good clearance between the plasma and the facing components is an essential feature of any plasma position and shape control system. Furthermore, plasma shape control is essential also for the heat flux control in the divertor region (more details can be found in [31]).

In this part of the thesis we show how the tokamak plasmas shape control actually appears to be a relevant application field in which the effectiveness of the proposed allocation solutions can be proved. In Chap. 7, in particular, we focus on the FTU tokamak, in which the horizontal position is controlled through two actuators. In order to provide FTU with an elongation control system, we combine the use of the two actuators adapting to this case the allocation technique described in Chap. 3.

In Chap. 8, we focus on the JET tokamak, in which the plasma position and shape are measured through a set of geometrical descriptors which are larger in number with respect to the number of PF coils available for the plasma shape control. In order to provide the JET control system with a saturation avoidance system, we implement an allocator block like the one proposed in Chap. 4. To assess the performance of the new integrated control system, many simulations are shown.
Chapter 7

Elongation regulation at
FTU

7.1 Introduction

The main nuclear fusion experiment in Italy is the Frascati Tokamak Upgrade (FTU) in Frascati. FTU (see [32]) is a medium size tokamak characterized by a high magnetic field. Its main goal is the study of radio-frequency (RF) auxiliary heating techniques.

In this chapter we employ and extend the dynamic allocation theory presented in Chap. 3 to guarantee asymptotic tracking of a prescribed plasma elongation in FTU. This task is hard to accomplish because it can only be achieved using the PF coil called F, a high bandwidth actuator needed to perform high performance horizontal plasma position regulation. Another actuator, the PF coil called V, is available for horizontal position regulation but its bandwidth is insufficient to suitably perform the horizontal position regulation task. Via the dynamic allocation technique it is possible to hierarchically achieve the two goals using the two actuators: the high priority (fast) goal is the horizontal position regulation task, while the low priority (slow) goal
is the elongation regulation. We present theoretical results supporting the proposed scheme, as well as simulations and experiments showing the effectiveness of the proposed solution.

Some preliminary results in the direction pursued here have appeared in [33] and [34]. The most mature version of this work appears in [35], while further experimental results are presented in [36].

### 7.2 Problem definition

The horizontal position control system at the Frascati Tokamak Upgrade (FTU) uses two magnetic actuators: the V coil (V standing for “vertical”, as it generates a vertical magnetic field which acts on the plasma horizontal position) and the F coil (F standing for “feedback”, as only this coil currently runs in feedback configuration). The magnetic field generated by the F coil influences both the plasma horizontal position and its elongation: in fact, there is a known quasi-static map which well approximates the relationship between the current flowing in the F coil and the elongation (see Sec. 7.4 for more details). Moreover, the two actuators’ power supply systems are characterized by different amplitude and rate saturation levels reported in Table 7.1.

In fact, the V amplifier can supply large currents, but it has strict current rate limits; on the other hand the F amplifier allows for steeper current profiles, but is more limited in the amplitude range.

<table>
<thead>
<tr>
<th>coil</th>
<th>Max current [kA]</th>
<th>Max current rate [kA/s]</th>
</tr>
</thead>
<tbody>
<tr>
<td>V</td>
<td>25.0</td>
<td>54</td>
</tr>
<tr>
<td>F</td>
<td>12.5</td>
<td>830</td>
</tr>
</tbody>
</table>

Table 7.1: Amplitude and rate saturation levels for the two actuators.

control system currently used at FTU ensures the regulation of the plasma horizontal position to the requested value, but does not care about its elongation. The horizontal position control (Fig. 7.3) con-
Figure 7.1: The magnetic field generated by the V coil at FTU. The view shows a cross section of the vacuum vessel and the V circuit. The plasma current and the V current are orthogonal to the section shown, with sign that changes from shot to shot, depending on the selected magnetic configuration and the consequent adjustable coils wiring.
Figure 7.2: The magnetic field generated by the F coil at FTU. The view shows a cross section of the vacuum vessel and the F circuit. The plasma current and the F current are orthogonal to the section shown, with sign that changes from shot to shot, depending on the selected magnetic configuration and the consequent adjustable coils wiring.

The reason why the V coil is not involved in the feedback regulation of the horizontal position is that it is too slow for this crucial task, due to its rate limits.

On the other hand, with this scheme in place, the actual current flowing in the F coil during the experiment depends on the PID controller output and therefore on the type of disturbances acting on the experimental system. In most experiments the experience of the physicists leads to a current of roughly 4 kA in the flattop phase, which induces a desirable elongation (typically around $1.03, 1.04$). However, any external disturbance that requires a significant action from the
PID controller (such as impurities, radiofrequency heating or similar events) causes a significant change in the steady-state current in the F coil and therefore an undesirable elongation. Elongations smaller than 1.03, because of the symmetries, make it difficult to identify certain plasma parameters from the magnetic measurements, thus compromising the plasma equilibrium reconstruction (namely the off-line computation of certain internal parameters of the plasma, often relevant for the physical results of the experiment). One of the many reasons why it is desirable to have an elongation regulation system is maintaining the elongation at sufficiently high values to avoid this reconstruction problem during most of the experimental pulse.

The only available actuator to design an elongation control for FTU is the F coil but it is not possible to completely rely on the V coil for the position control, because of its rate saturation limits. Nevertheless the slow V coil could be used to slowly take over the F coil effort and make the F coil available for the lower priority task of elongation control which does not require the same bandwidth. So a coordinated action of the two actuators is needed, but classical linear control techniques cannot be used within this setting to guarantee elongation control without resulting in a perturbation for the horizontal position control loop. Moreover with a linear approach the only way to avoid the rate saturation of the V coil is to design a slow enough control, in order to remain inside the bounds also in the worst case, thus resulting in a too conservative solution.

In this chapter we employ the dynamic allocation scheme of Chap. 3 to combine the action of the V and F coils. From the viewpoint of the horizontal position, the two coils are redundant and can be used to achieve the horizontal regulation goal as a primary task and regulation of the current flowing in the F coil as a secondary task, carried out without perturbing the position loop at all. This secondary task
results in ensuring a desired elongation because of the direct relationship between the achieved elongation and the current flowing in the F coil. In particular, once the current in the F coil can be suitably regulated, it is possible to achieve a prescribed desired elongation by suitably inverting the known affine curve relating the F coil current to the corresponding plasma elongation. Since elongation regulation is a low priority (low bandwidth) task, very large disturbances would then alter the elongation but only temporarily. In most practical cases, the elongation regulation would be satisfactory.

The chapter is organized as follows: in Sec. 7.3 we describe the allocation scheme and give theoretical results about the $i_F$ current regulation task, presenting simulations and experimental results, in Sec. 7.4 we describe the adaptation of the allocation scheme for achieving elongation regulation, giving theoretical results and showing simulations and experimental results.

### 7.3 Current allocation

For our purposes, the plasma horizontal position behaviour can be described by a simple linear model. The model we use for designing our control system was first introduced in [37]. According to it, the measured output $\Delta \Psi$ can be computed from the currents $i_V$, $i_F$ and $i_P$ through the transfer function

$$\Delta \Psi(s) = F_P(s) \begin{bmatrix} k_V & k_F & k_P \end{bmatrix} \begin{bmatrix} i_V(s) \\ i_F(s) \\ i_P(s) \end{bmatrix}$$

(7.1)

with

$$F_P(s) = \frac{s}{\tau s + 1}.$$  

(7.3)
where $\tau$ represents the main time constant of the plasma+coils dynamics, while the coefficients $k_V$, $k_F$ and $k_P$ quantify the steady-state effect of each current on the plasma horizontal position.

Despite the simplicity of the model, it reproduces accurately enough the measured data during the flat-top phase of the experiments.

In order to apply the results shown in Chap. 3, we need the plant to be described in state space form. A possible state space minimal realization for our transfer function is given by:

$$
\begin{align*}
\dot{x} &= Ax + Bu + B_d d, \\
y &= Cx + Du + D_d d,
\end{align*}
$$

(7.4)

where the control input $u = \begin{bmatrix} i_V \\ i_F \end{bmatrix}$ corresponds to the currents requested, respectively, from the V and F coils, the disturbance $d = i_P$ corresponds to the plasma current and the plant output $y = \Delta \Psi$ is our indirect measure of the radial position error. The system matrices in model (7.4) are given by:

$$
\begin{bmatrix}
A & B & B_d \\
C & D & D_d
\end{bmatrix} = \frac{1}{\tau} \begin{bmatrix}
-1 & k_V & k_F & k_P \\
-1 & k_V & k_F & k_P
\end{bmatrix}.
$$

(7.5)

The horizontal position control scheme currently used at FTU is represented in Fig. 7.3. It consists in a feedforward action on both the inputs $i_V$ and $i_F$ plus a feedback correction only on $i_F$, obtained from a PID control. For our purposes the overall controller can be represented as another generic linear time invariant system of the form:

$$
\begin{align*}
\dot{x}_c &= A_c x_c + B_c u_c + B_r r, \\
y_c &= C_c x_c + D_c u_c + D_r r,
\end{align*}
$$

(7.6)

where $A_c, B_c, B_r, C_c, D_c, D_r$ are suitable constant matrices, $r$ represents the reference and $y_c = \begin{bmatrix} i_{Vc} \\ i_{Fc} \end{bmatrix}$ is the control signal for the actuators. The two systems, the controller (7.6) and the plant (7.4), are connected to each other as follows:

$$
u_c = y, \quad u = y_c,
$$

(7.7) (7.8)
namely the control is in feedback from the $\Delta \Psi$ measurement and the PF currents are driven by the controller (7.6). The whole horizontal position control scheme is well posed and assures asymptotic stability, so it actually satisfies Assumption 3.1.

Figure 7.3: The current horizontal position regulation scheme. Both the $V$ and $F$ actuators contribute to this task: the $V$ coil is driven by a feedforward action, while the $F$ coil is driven by both a feedforward action and a feedback PID action from the measurement $\Delta \Psi$.

Considering $\Delta \Psi$ as the only output, i.e. from the sole point of view of the horizontal position regulation, the plant results to be strongly input redundant (see Chap. 3), namely it is possible to find a matrix $B_\perp$ such that

$$\text{Im}(B_\perp) = \text{Ker} \left( \begin{bmatrix} B \\ D \end{bmatrix} \right) \neq \{0\},$$

therefore representing a basis of the kernel in (7.9).

By adapting the approach in Chap. 3 to our needs for this application, a dynamic allocator can be designed as follows, as represented in Fig. 7.5

$$g = -\rho B_\perp^T W(t,u_a)(u_a - u_r),$$

$$\dot{\delta} = \bar{\sigma}_{MR}(\delta,g),$$

$$y_a = B_\perp \delta,$$
where $u_a$ is the input to the allocator block and $u_r$ is the desired reallocated value for this input. Moreover $\bar{\sigma}_{MR}(\cdot, \cdot)$ is a set-valued map defined as

$$\bar{\sigma}_{MR}(x, u) = \begin{cases} 
\sigma_R(u), & |x| < M, \\
\overline{\sigma_R(\{\sigma_R(u), -R \text{sgn}(x)\})}, & |x| = M, \\
-R \text{sgn}(x), & |x| > M,
\end{cases} \quad (7.11)$$

where $\sigma_R(x) = \text{sgn}(x) \min(|x|, R)$ is the saturation function, $\overline{\sigma}(\cdot)$ is the convex hull of a set, $\rho$, $M$ and $R$ are positive scalars and $W(\cdot)$ is a weight square matrix described below.

Equation (7.10b) represents a saturated integrator with rate saturation (see Fig. 7.5) namely an integrator whose state can only grow up to a certain saturation level with a limited velocity. The interconnection between the allocator (7.10) and the closed-loop, as shown in
Fig. 7.4, is described via the equations:

\[ u_a = u = \begin{bmatrix} i_V \\ i_F \end{bmatrix}, \quad (7.12a) \]

\[ u_r = \begin{bmatrix} i_{V_r} \\ i_{F_r} \end{bmatrix}, \quad (7.12b) \]

\[ u = y_c + y_a, \quad (7.13) \]

namely the input signal to the allocator \( u_a \) is the same one that enters the plant: the PF currents, while the reference signal \( u_r \) represents their desired values requested from the allocator. Equations (7.10c) and (7.13) together ensure that the allocator contribution belongs to the input space kernel so that no changes to the plant input \( u \) will be visible on the plant output \( \Delta \Psi \) as compared to the configuration without the allocator. Since the redundancy is strong, this is true both at the steady-state and during the transients.

The allocator state \( \delta \) specifies the operating point in this kernel subspace. We notice that \( \delta \) is the output of a saturated integrator with saturation level equal to \( M \): this element allows the designer to limit in amplitude the allocator contribution, for example for safety reasons.

The parameter \( R \) represents instead a rate saturation level for the allocator state \( \delta \) and consequently for the allocator contribution \( y_a \). This parameter can be tuned in order to ensure the feasibility of the requested currents with respect to the power supplies rate limits.

When we are working in linear conditions, namely the two saturations are not active, the parameter \( \rho \) specifies the allocator gain, i.e. its convergence speed. Due to the strong input redundancy of the plant the value for \( \rho \) can be adjusted theoretically as fast as desired. Actually some restrictions on \( \rho \) are necessary to ensure stability, as confirmed by the experiments of section 7.4.2, because of unmodelled dynamics and other modeling approximations. See, in particular, the
oscillations in Fig. 7.20 not predicted by the simulation of Fig. 7.17.

A possible choice for $B_\perp$ is:

$$B_\perp = \begin{bmatrix} 1 \\ b \end{bmatrix}, \quad (7.14)$$

with $b = -\frac{k_V}{k_F}$. This particular choice is convenient because the allocator state $\delta$ corresponds to the $i_V$ current variation $\Delta i_V$ so that the parameters $M$ and $R$ are, respectively, a magnitude and a rate limit on the slow actuator contribution.

The weighting matrix is chosen as

$$W(t, u_a) = \begin{bmatrix} w_V(t, u_a) & 0 \\ 0 & w_F(t, u_a) \end{bmatrix} \geq 0 \quad (7.15)$$

with $\max(w_V, w_F) > 0$ for all pairs $(t, u_a)$. For constant references, if a constant matrix $W = \begin{bmatrix} w_1 & 0 \\ 0 & w_2 \end{bmatrix} \geq 0$ is used, the steady-state reallocated input is the one that minimizes the cost function $J(t, u_a) = (u_a - u_r)^T W (u_a - u_r)$ subject to the constraint $u_a = y_c + B_\perp \delta$, so, using the vector $(w_1, w_2)$, it is possible to decide (see Chap. 3) which error to penalize more; in particular if the weight of one input is equal to zero, then the other input will asymptotically reach the reference value.

The goal in this application is to make the $i_F$ current follow a desired reference signal, while there is no interest in making the $i_V$ current follow a particular one. On the other hand it is possible to exploit the weight function in order to obtain a smooth activation mechanism for the allocator.

If we choose $i_{Vr} = i_{Vc}$ as reference for the $i_V$ current, when $w_F(t, u) = 0$ the allocator effort is directed towards minimizing its own contribution and asymptotically shut down; on the other hand when $w_V(t, u) = 0$ the allocator effort is directed towards regulating the
$i_F$ current to the reference value. In summary taking

$$W(t, u_a) = \begin{bmatrix} 1 - w_F(t, u_a) & 0 \\ 0 & w_F(t, u_a) \end{bmatrix}$$

(7.16)

with $0 \leq w_F(t, u_a) \leq 1$ for all pairs $(t, u_a)$, allows to choose when the allocator must track the $i_F$ current reference and when it must shut down: this can be chosen depending both on time and on the input, if the allocator must give no contribution at certain times or, respectively, there are some values of $u_a$ the allocator is not allowed to generate.

A possible choice for $w_F(t, u)$ is:

$$w_F(t, u) = \gamma(u_2)r_{ON}(t)$$

(7.17)

where $r_{ON}(t)$ is an activation external signal to turn on and off the allocator at different times, satisfying $0 \leq r_{ON}(t) \leq 1$ for all times, while $\gamma(\cdot)$ can be chosen as the following trapezoidal function

$$\gamma(x) = \begin{cases} 
0 & x \leq x_1, \\
\frac{x - x_1}{x_2 - x_1} & x_1 < x \leq x_2, \\
1 & x_2 < x \leq x_3, \\
\frac{-x + x_4}{x_4 - x_3} & x_3 < x \leq x_4, \\
0 & x_4 < x
\end{cases}$$

(7.18)

with suitable choices for the parameters $x_k$, $k = 1, ..., 4$ and with $x_2 < 0 < x_3$. These parameters can be selected in such a way that the allocator remains fully active when the $i_F$ current is far from its saturation levels, namely in the range $[x_2, x_3]$, the allocator is fully inactive when the $i_F$ current is outside from the range $[x_1, x_4]$, while it gradually shuts down when the $i_F$ current is in the intermediate ranges $[x_1, x_2]$ and $[x_3, x_4]$.

By designing the allocator as described so far, we can assure the whole closed-loop system with the allocator to have some desirable
7.3 Current allocation

properties. We prove these properties in the following theorem.

**Theorem 7.1** Consider the closed-loop system with input allocation given by (7.4), (7.6), (7.7), (7.10), (7.12) and (7.13). If Assumption 3.1 is verified, the system has the following properties:

a) Given any initial condition $x(0), x_c(0), \delta(0)$ and any selection of $r(\cdot), d(\cdot), u_r(\cdot)$, the plant output $y$ of the system and that of the system without input allocation (given by (7.4), (7.6), (7.7), (7.8)) coincide at all times.

b) The allocator state response is limited in rate, namely $|\dot{\delta}(t)| \leq R$ at almost all times. Moreover, the set $\mathcal{M} := \{\delta : |\delta| \leq M\}$ is forward invariant and any initial state $\delta_0$ outside $\mathcal{M}$ converges to $\mathcal{M}$ after the finite time $\frac{|\delta_0| - M}{R}$.

Moreover, if $r_{ON}(t) = 1, \forall t$, then:

c) if $x_2 < i_{F_c} + bM \leq i_{F_c} - bM < x_3$ then the allocator has a globally asymptotically stable equilibrium point in $\bar{\delta} = \sigma_M \left( \frac{i_{F_c} - i_{F_r}}{b} \right)$. Moreover if $|i_{F_c} - i_{F_r}| < |b|M$ then $\bar{\delta}$ is locally exponentially stable and the current $i_F$ converges to the reference value $i_{F_r}$; otherwise $\bar{\delta}$ is reached in finite time;

d) if $i_{F_c} - bM < x_1$ or $i_{F_c} + bM > x_4$ then the allocator has a globally asymptotically, locally exponentially stable equilibrium point in $\bar{\delta} = 0$ and the current $i_F$ converges to the linear controller output $i_{F_c}$.

Finally, if $r_{ON}(t) = 0, \forall t$, then:

e) the allocator has a globally asymptotically, locally exponentially stable equilibrium point in $\bar{\delta} = 0$ and the current $i_F$ converges to the linear controller output $i_{F_c}$. 

Proof. From the choice (7.9) of the matrix $B_\perp$ it follows that the allocated closed-loop system is equivalent to the cascade of two systems: the non allocated closed-loop system and the allocator system (7.10). So the output $y$ of the non allocated closed-loop system is completely independent from the allocator behaviour, proving point (a).

Point (b) directly follows from the definition of the set-valued map (7.11), because for every pair of values of the arguments, all the elements $y \in \bar{\sigma}_{MR}(x,u)$ satisfy $|y| \leq R$.

To prove point (c), two situations must be considered. If $|i_{F_r} - i_{F_c}| < |b|M$ then the equilibrium $\bar{\delta}$ is given by $\bar{\delta} = \frac{i_{F_r} - i_{F_c}}{b}$ and the function $V(\delta) = \frac{1}{2} (\delta - \bar{\delta})^2$ is a Lyapunov function. Indeed its Lyapunov derivative is given by:

$$
\dot{V} \in \begin{cases} 
\{(\delta - \bar{\delta}) \sigma_R(-\rho b^2 (\delta - \bar{\delta}))\}, & |\delta - \bar{\delta}| < M, \\
\Co\{(\delta - \bar{\delta}) \sigma_R(-\rho b^2 (\delta - \bar{\delta}))\}, & |\delta - \bar{\delta}| = M, \\
\{(\delta - \bar{\delta}) (R \sgn(\delta))\}, & |\delta - \bar{\delta}| > M,
\end{cases} \tag{7.19}
$$

with $\dot{M} := \min\{M, \frac{R}{\rho b^2}\}$ and it results to be negative definite. Finally if $|\bar{\delta}| < M$ then in a neighborhood of the equilibrium point $\bar{\delta}$, the condition $\dot{V} = 2\rho V$ is verified, so that the stability is locally exponential. Moreover note that when $\delta = \bar{\delta}$ the current $i_F$ results to be $i_F = b\delta + i_{F_c} = b\frac{i_{F_r} - i_{F_c}}{b} + i_{F_c} = i_{F_r}$.

In the second situation, if $i_{F_r} - i_{F_c} \geq |b|M$ the equilibrium point $\bar{\delta}$ is given by $\bar{\delta} = M$ (while it is $\bar{\delta} = -M$ if $i_{F_r} - i_{F_c} \leq -|b|M$, in which case the reasoning is the same). Consider the time derivative of the
state, given by

\[
\dot{\delta} \in \begin{cases}
\{\sigma_R \left( -\rho \delta^2 \left( \delta - \frac{i_{Fb} - i_{Fc}}{b} \right) \right) \}, & \delta < M, \\
\text{Co}\{\sigma_R \left( -\rho \delta^2 \left( \delta - \frac{i_{Fb} - i_{Fc}}{b} \right) \right) \}, & \delta = M, \\
\{-R \text{sgn} (\delta)\}, & \delta > M.
\end{cases}
\] (7.21)

and note that there exist \( c > 0 \) such that for \( \delta < M \) it holds that \( \dot{\delta} \geq c > 0 \), while for \( \delta > M \) it holds that \( \dot{\delta} \leq -R < 0 \). So the state converges in finite time to \( \bar{\delta} = M \).

Under the assumptions of point (d), the function \( V(\delta) = \frac{1}{2} \delta^2 \) is a Lyapunov function. Indeed the Lyapunov derivative is given by

\[
\dot{V} \in \begin{cases}
\{\delta \sigma_R (-\rho \delta)\}, & \vert \delta \vert < \hat{M}, \\
\text{Co}\{\delta \sigma_R (-\rho \delta), \delta \left( -R \text{sgn} (\delta) \right)\}, & \vert \delta \vert = \hat{M}, \\
\{\delta \left( -R \text{sgn} (\delta) \right)\}, & \vert \delta \vert > \hat{M}.
\end{cases}
\] (7.22)

which results to be negative definite. Moreover in a neighborhood of the origin we have \( \dot{V} = 2 \rho V \), so that the stability is locally exponential. The proof of point (e) parallels to that of point (d).

\[\square\]

In particular

- \( (a) \) ensures that the allocation is transparent with respect to the system’s output and state responses;
- \( (b) \) ensures that the allocation is performed with a bounded rate which can be chosen by the designer. Thus we can avoid the allocator to request current from the V coil at a rate over its limit;
- \( (c),(d) \) and \( (e) \) ensure that the current \( i_F \) converges to a desirable value. The desirable value depends on the current operating region of the system and on the activation signal \( r_{ON}(t) \).
establishing whether $i_F$ should be regulated primarily for performance or for safety.

### 7.3.1 Simulations

In this section some simulation results are presented to illustrate how the allocator works and how the different parameters can be tuned. In all the simulations, the parameters of model (7.4) are selected as: $k_P = -1.12e^{-8}$, $k_V = -1.18e^{-6}$, $k_F = -2.75e^{-7}$, $\tau = 8e^{-3}$. To reproduce the experimental scenario of FTU, the PID controller is implemented by the discrete-time transfer function

\[
\frac{i_{F_{PID}}(z)}{\Delta \Psi(z)} = -k_p - k_i \frac{T_s z + 1}{2 z - 1} - k_d \frac{z - 1}{(\tau_d + T_s)z - \tau_d} \tag{7.24}
\]

where $T_s = 5e^{-4}$ is the sampling time, $\tau_d = 4e^{-3}$. According to the configuration of the FTU control system, the PID gains are not constant, but first ramp up from zero to the steady-state values $k_p = 7.5e^3$, $k_i = 1e^6$ and $k_d = 22.5$ in the time intervals $[0, 0.05s]$, $[0, 0.16s]$ and $[0, 0.02s]$, respectively; then they remain constant and finally ramp down to zero in the time intervals $[1.6, 2s]$, $[1.6, 2s]$ and $[1.6, 1.8s]$, respectively. The overall linear controller action is given by

\[
y_c = \begin{bmatrix} i_{VF} \\ i_{FF} + i_{F_{PID}} \end{bmatrix}. \tag{7.25}
\]

The activation signal $r_{ON}(\cdot)$ is chosen as a trapezoidal shape to activate the allocator at $0.4s$ when the ramp-up phase is finished and smoothly shut down during the time window $[1.4s, 1.5s]$ (see the upper plot in Fig. 7.6). Thanks to this the allocator is activated just during the flat-top phase, when the PID gains are constant at their flat-top values. So for our purposes the controller can be considered as a common PID and the results proved in Sec. 7.3 and 7.4 guarantee the stability of the closed-loop.

In the first simulation, as shown by the dash-dotted curves in the middle plots of Fig. 7.6 the input signals for the closed-loop model...
\( i_{VF_F} \), \( i_{F_F} \), and \( i_P \) have a trapezoidal shape: \( i_{VF_F} \) ramps up from 0A and reaches 4938.9A at time 0.2585s, then ramps up to 4945A at time 1.5s, and finally ramps down to 0A at time 1.8s; \( i_{F_F} \) jumps to 2000A at time 0s, then ramps up to 2500A at time 1.5s, and finally ramps down to 0A at time 1.7s; \( i_P \) ramps from 0s up to \(-5e^5\)A at time 0.148s, then is constant until time 1.5s, and finally ramps down to 0A at time 1.8s. The reference \( i_{F_r} \) driving the allocator is the piecewise constant signal corresponding to the dashed curve in the middle-top plot of Fig. 7.6.

The same simulation is repeated with the allocator working in current mode, i.e. with a \( i_F \) current reference, and without the allocator. In Fig. 7.6 the plant inputs and outputs are shown. The two simulations appear the same up to time 0.4s when the allocator is turned on and starts modifying the inputs in order to regulate the current \( i_F \) to the reference value. It is evident that, while the input signals are different in the two simulations, the plant outputs \( \Delta \Psi \) are perfectly superimposed in the two simulations, showing that the allocator is invisible from the plant output.

When large instantaneous excursions are requested from the allocator, for example at time 0.8s, the rate saturation effect is evident in the first part of the transient, while in the second part of the transient, when the difference between \( i_F \) and \( i_{F_r} \) is smaller, the exponential convergence appears. All the \( i_F \) variations requested by the allocator, can also be seen on the \( i_V \) current request, suitably scaled.

In order to observe how the shut down mechanism works, the same simulation is repeated in Fig. 7.7 with more stringent values of the weight function parameters, namely \( x_3 = 3500A \) and \( x_4 = 3700A \), and with a current reference \( i_{F_r} \) which requests the allocator to bring the current \( i_F \) in a forbidden region. When \( i_F \) passes through the boundary zone \([x_3, x_4] \), the weight \( w_F \) decreases so that the allocator is almost shut off. As a consequence, in that part of the simulation,
Figure 7.6: Closed-loop simulation with (solid) and without (dash-dotted) the allocator working in current mode. When the allocator is active ($r_{ON} = 1$), the plant inputs are changed in order to regulate the current $i_F$ to the reference value (dashed) without changing the plant output $\Delta \Psi$. The reference signal is not tracked. After time 1.15s, the allocator is requested to drive the current $i_F$ back inside the allowed interval and the reference signal is tracked again. At time 1.4s, since we are close to the end of the pulse, the weight $w_F$ is decreased by the decreasing activation signal $r_{ON}(\cdot)$ for the final shut-off and this limits again the allocator’s authority.
7.3 Current allocation

We run some closed-loop simulations of (7.4) and (7.10).

Fig. 7.8 represents two other simulations. In the first simulation (light dash-dotted) the control system is run without the allocator (namely the darkest block in Fig. 7.4 is disconnected). In the second simulation (dark solid) the allocator is connected and the reference signal $i_F$ is selected as the dashed line in the upper plot. The constant $k$ in (7.10) has been selected as $k = 1$ to induce a desirable small signals behavior. The saturation level of $\sigma_R(\cdot)$ has been selected as $R = 3800\,\text{A/s}$ to induce a desirable large signal slope compatible with the limits in (7.1) and the saturation level of $\sigma_M(\cdot)$ is $M = 2000\,\text{A}$ so
that the allocator can only use a limited amount of input authority. From the simulation it appears that the allocator successfully guarantees graceful and slow convergence of the $F$ current to the desired value, while no difference is seen at the plant output $\Delta \Psi$ (the dark and light curves are on top of each other there). The lowest plot of the figure shows the plasma current value, taken from experimental data, which works as a disturbance, causing the perturbation in the horizontal position around time 0.9s.
7.3 Current allocation

7.3.2 Experiments

The dynamic allocation algorithm has been implemented in the FTU control system and tested during a series of experimental shots. Some experiments are presented in this section showing the allocator working in current mode.

Two experiments are presented with the allocator working in current mode: the allocator tracks a reference signal for $i_F$, modifying the current $i_V$ in order to maintain the horizontal plasma position unchanged. The two shots have different $i_F$ current references and different operating conditions. Instead the allocator parameters are the same in both the experiments: the allocator gain is $\rho = 7$, while

![Figure 7.9: Shot 31725: the current $i_F$ (upper solid) tracks the reference (upper dashed), while the current $i_V$ (middle solid) changes with respect to its preprogrammed feedforward value (middle dash-dotted) in order to leave $\Delta \Psi$ (lower solid) unchanged.](image)
a saturation value of 5000\,A/s is imposed on the allocator state rate \( \dot{\delta} \) (which corresponds to the maximum \( i_V \) current rate, thanks to the specific choice of \( B_\perp \) in (7.14)). Mimicking the top plot of Fig. 7.6, the activation signal \( r_{ON} \) is zero up to time 0.4\,s, one up to time 1.4\,s, then linearly ramps down to zero in the time window from 1.4\,s to 1.5\,s. This choice ensures that the allocator is fully active only during the flat top phase and then smoothly shuts down.

Shot number 31725 in Fig. 7.9 reproduces experimentally the same test made in the simulation of Fig. 7.6, namely the same input signals are used: a piecewise constant reference is requested for the current \( i_F \) jumping from 3000\,A to 2000\,A and again to 3000\,A (see top plot). From time 0.4\,s, when the allocator is activated, the current \( i_V \) (solid curve in the middle plot) is moved away from its preprogrammed feedforward value (dash-dotted curve in the middle plot) in order to track the \( i_F \) current reference value of 3000\,A. The rate saturation effect is evident in the solid \( V \) current trace of the middle plot around times 0.8\,s and 1.1\,s, when the reference instantaneously jumps from 3000\,A to 2000\,A and vice versa. The rate saturation inside the allocator prevents it from generating an infeasible request from the power supply. Despite the deviations of the currents in the \( F \) and \( V \) coils from the preprogrammed feedforward values, the output signal \( \Delta \Psi \) (lower plot) is unaffected by the large input excursions.

Shot 31855 is shown in Fig. 7.10, where the preprogrammed feedforward \( i_{VF_F} \) current has a peculiar profile, presenting a bump between times 0.3\,s and 0.9\,s. The reference is again piecewise constant, but jumping from \(-2000\,A\) to 0\,A and finally to \(-3000\,A\) (see top plot). When the allocator is activated at \( t = 0.4\,s \), the current \( i_F \) ramps down to the reference value at the maximum admissible rate and reaches \(-2000\,A\) just in time to ramp up towards the new 0\,A reference value, where it remains up to time 0.85\,s, when it begins to ramp down again to reach \(-3000\,A\). This second ramp down starts before the \( i_F \) reference changes due to the variations of the preprogrammed feedforward
Figure 7.10: Shot 31855: the current $i_F$ (upper solid) follows the reference (upper dashed), while the current $i_V$ (middle solid) changes with respect to its preprogrammed feedforward value (middle dash-dotted) in order to keep $\Delta \Psi$ (lower solid) unchanged.

In Fig. 7.11 two identical experiments are compared: shots number 31724 (dark solid) and 31710 (light dash-dotted) where the allocator is active and inactive, respectively. The allocator works in current mode with a constant reference of 3000 A for the F current (dashed line in the upper plot). It can be seen that the F current (top plot) follows the constant reference, while the V current is modified accordingly to $i_{FF}$ and $i_{VF}$ currents (see the dash-dotted curves in the two upper plots). Also in this second case of Fig. 7.10 the output $\Delta \Psi$ is not affected by the allocator. Indeed note that the two peaks at times 0.3s and 0.9s are not caused by the allocator but by the ramps in the preprogrammed feedforward $i_{VF}$ current signal representing a disturbance for the PID controller.
Figure 7.11: Comparison between shots number 31710 (light dash-dotted) without the allocator and 31724 (dark solid) with the allocator.

make the allocator action invisible at the output.

The oscillations on the V coils in the middle plot of Fig. 7.11 reveals that the allocator action is too aggressive. Therefore, in Fig. 7.12 the same shot of the previous figure is compared to another shot number 31726 where a smaller value for the rate saturation limit $R$ has been selected ($R$ was selected as 5000A/s in shot number 31724 and as 3800A/s in shot number 31726). Reduced oscillations are seen after the saturation level reduction as seen in the dark solid curve of the figure.
7.4 Elongation regulation

An important goal of this application is to obtain a slow regulation of the plasma elongation $\kappa$, defined as the ratio between the vertical and horizontal plasma length. This can be done using the allocation scheme presented in the previous section thanks to a quasi-static relationship between the elongation $\kappa$ and the current $i_F$. Since the F winding generates a magnetic field that is not perfectly vertical then, in addition to moving the plasma horizontally, it produces a slight compression of the plasma, thus affecting its elongation. In particular, the steady-state elongation $\kappa$ induced by the F coils current can be computed, also based on the plasma current $i_P$ as

$$\kappa = f(i_F, i_P)$$  \hspace{1cm} (7.26)

which can be approximated by the map

$$\hat{\kappa} = \hat{f}(i_F) = \kappa_0 - \kappa_1 \frac{i_F}{i_P},$$  \hspace{1cm} (7.27)

which is affine in $i_F$, while the current $i_P$ can be seen as a time varying parameter. The numerical values of the parameters $\kappa_0$ and $\kappa_1$ will be provided elsewhere as a function of the magnetic field $B$ and the plasma current $i_P$. 

Figure 7.12: Comparison between shots number 31724 (light dash-dotted) with $R = 5000\, \text{A/s}$ and 31726 (dark solid) with $R = 3800\, \text{A/s}$.
\( \kappa_1 \) have been estimated by a least squares fit of the experimental data, reported in Fig. 7.13. In particular they correspond to \( \kappa_0 = 1.03 \) and \( \kappa_1 = -4.61 \).

This formula suitably describes elongation behaviour for typical FTU plasmas. A graphical representation of the relationship between \( \kappa \) and \( i_F \) is shown in Fig. 7.13. The static map is affine and depends on \( i_P \) so that a family of curves is shown in Fig. 7.14, parametrized by \( i_P \).

Since \( i_P \) is measured in real-time, the map \( \hat{f}(\cdot) \) in (7.27) from \( i_F \) to \( \kappa \) can be inverted so that given a certain elongation \( \kappa \), the current \( i_F \) enforcing that elongation is computed as

\[
i_F = f^{-1}(\kappa)
\]  

which is approximated by

\[
\hat{i}_F = \hat{f}^{-1}(\kappa) = \frac{i_P}{\kappa_1}(\kappa - \kappa_0).
\]
Finally the allocator inputs are chosen as

\[
\begin{align*}
  u_a &= \begin{bmatrix} i_V \\ \hat{i}_F \end{bmatrix}, \\
  u_r &= \begin{bmatrix} i_{V_r} \\ \hat{i}_{F_r} \end{bmatrix},
\end{align*}
\]  

(7.30a)

\[
\hat{i}_F = \hat{f}^{-1}(\kappa), \quad \hat{i}_{F_r} = \hat{f}^{-1}(\kappa_r).
\]  

(7.30b)

where \( \hat{i}_F \) and \( \hat{i}_{F_r} \) are computed using the approximate inverse of \( f(\cdot) \) according to (7.29), using, respectively, the elongation measurement which is available in real-time and the elongation reference \( \kappa_r \). The overall closed-loop is in feedback from the elongation measurement signal and is represented in Fig. 7.15. For this scheme the following result can be proven.

**Theorem 7.2** Consider the modified control system given by (7.4), (7.6), (7.7), (7.10), (7.13), (7.26), (7.30). If in this system \( \hat{i}_F \) asymptotically converges to \( \hat{i}_{F_r} \), then also \( \kappa \) asymptotically converges to \( \kappa_r \).

**Proof.** If \( i_F = i_{F_r} \), from the choice (7.30b) it follows that \( \hat{f}^{-1}(\kappa) = \hat{f}^{-1}(\kappa_r) \). Since the function \( \hat{f}(\cdot) \) is invertible, then this is true if and only if \( \hat{f}(\hat{f}^{-1}(\kappa)) = \hat{f}(\hat{f}^{-1}(\kappa_r)) \), i.e. \( \kappa = \kappa_r \). \( \square \)
Remark 7.1 The convergence property assumed in Theorem 7.2 trivially holds if $\dot{f} = f$ and $i_{F_c}, i_{F_r}$ satisfy the conditions in case (c) of Theorem 7.1. Then Theorem 7.2 shows that the convergence of $\kappa$ to $\kappa_r$ is still ensured if $\dot{f}$ is just an approximation of $f$, as long as convergence of $\dot{i}_F$ to $\dot{i}_{F_r}$ is preserved (which, by standard reasonings about small perturbations is true, provided that $\dot{f}$ is sufficiently close to $f$). This robustness property is similar to what is achieved in output tracking when an internal model of the reference is used, i.e. tracking is robust as long as stability is preserved.

Figure 7.15: Block diagram of the allocation scheme for elongation regulation.

7.4.1 Simulations

A first simulation with the elongation regulator disabled is shown in Fig. 7.16 by the dotted curves: the currents $i_F$ and $i_V$ just follow their preprogrammed values, resulting in a constant elongation. The input signals $i_{V_{FF}}, i_{F_{PP}}$ and $i_P$ have the following trapezoidal shapes: $i_{V_{FF}}$ ramps from 0 up to $-4940$ at time 0.3, then sits at $-4945$ up to time 1.5, and finally ramps down to 0 at time 1.8; $i_{F_{PP}}$ ramps from 0 up to $-1500$ at time 0.14, then sits at $-1500$ up to time 1.5, and finally ramps down to 0 at time 1.7; $i_P$ ramps from 0 up to $-5e^5$ at time 0.25, then sits at $-5e^5$ up to time 1.5, and finally ramps down to 0 at time 1.8.
7.4 Elongation regulation

The results of a second simulation are represented by the dash-dotted curves: in this case the elongation regulation is active with $R = 3000 \text{As}^{-1}$. The elongation reference, a piecewise constant signal (upper dashed), is translated to a reference for the current $i_F$ (middle-top dashed); while the current $i_F$ follows the reference, the current $i_V$ changes in order to maintain $\Delta \Psi = 0 \text{Wb}$. In all the curves the rate saturation is evident, due to the low value of $R$.

The same simulation is repeated with $R = 15000 \text{As}^{-1}$ (solid curves). In this case the rate saturation effect is smaller and the regulation results to be faster. The same simulation is repeated with different values of $\rho$, respectively $\rho = 7$, $\rho = 1$ and $\rho = 0.5$, to show how this affects the allocator convergence speed (see Fig. 7.17). It is evident that increasing $\rho$ the convergence speed increases too; at the same time, the rate saturation effect becomes more evident.

7.4.2 Experiments

Some experimental shots have been done to test the performance and the limitations of the implemented elongation control. Some experiments have also been done to find the best tuning of the free parameters.

In shots 31970, 31971 and 32963, shown in Fig. 7.18, the same tests made in the simulations of Fig. 7.16 are experimentally reproduced. In shot 31970 the elongation control is not active and so the current $i_V$ (dotted curve in the middle-bottom plot) just follows the preprogrammed value. Differently from the simulation, the elongation does not assume a constant value but varies in an uncontrolled way, due to the variations in the currents $i_F$ and $i_P$. Shot 31971 is an identical experiment with the elongation control active, reproducing the second simulation. The elongation (dash-dotted curve in the top plot) follows the reference value almost like in the simulation. At the same time the position error $\Delta \psi$ (dash-dotted curve in the bottom plot) is maintained almost equal to zero. The same experiment is also repeated
Figure 7.16: Closed-loop simulations with the allocator disabled (dotted) and with the allocator enabled with, respectively, $R = 3000\text{As}^{-1}$ (dash-dotted) and $R = 15000\text{As}^{-1}$ (solid). For each simulation the elongation (upper plot), the current $i_F$ (upper middle plot), the current $i_V$ (lower middle plot) and $\Delta \Psi$ (lower plot) are shown. The dashed line in the upper plot represents the elongation reference, while the dashed line in the upper-middle plot represents the corresponding current reference for the $F$ coil.

In shot 32963 with $R = 15000\text{As}^{-1}$, which imposes the maximum rate for the allocator contribution, reproducing the third simulation. Also in this case the elongation (solid curve in the top plot) follows the reference value almost like in the simulation. On the other hand,
7.4 Elongation regulation

Simulations with different values of $\rho$

Figure 7.17: Closed-loop simulations with the allocator working in elongation mode with different values of the parameter $\rho$: $\rho = 7$ (solid), $\rho = 1$ (dash-dotted), $\rho = 0.5$ (dashed). The greater the value of $\rho$, the faster the convergence to the reference value (upper light dashed) and the more the rate saturation effect is evident.

while in the simulation the position error $\Delta \psi$ is not affected by the allocator action, in the experiment we can see that the position control is a bit deteriorated when a too large variation is requested for the elongation, like at time 0.8s. It’s also interesting to note that both in shot 31971 and 32963 the currents $i_F$ and $i_V$, respectively, do not assume precisely the same values assumed in the simulations due to the presence of disturbances and approximations in the model; nevertheless the controlled variables, $\kappa$ and $\Delta \psi$, are regulated to the desired values.

A peculiar experiment, shot 32969, has also been done to verify whether the elongation regulator also works with elongation values smaller than 1, i.e. with plasmas with a horizontally squeezed cross section shape. This can be useful, for example, for future applications in the temperature control for the liquid lithium limiter developed at FTU. The elongation reference signal is piecewise constant and jumps
Figure 7.18: Shot 31970 with the allocator disabled (dotted), shot 31971 with the allocator enabled and \( R = 3000 \text{As}^{-1} \) (dash-dotted) and shot 32963 with \( R = 15000 \text{As}^{-1} \) (solid). For each shot the elongation (upper plot), the current \( i_F \) (upper-middle plot), the current \( i_V \) (lower-middle plot) and \( \Delta \Psi \) (lower plot) are shown.

from 1.04 to 1.00, then to 0.97 and finally to 1.02. The rate saturation value was set as \( R = 15000 \text{As}^{-1} \). From Fig. 7.19 it can be seen that the elongation is well regulated to the desired values, namely there are no problems with elongations smaller than 1, as it can be seen in the time interval \([0.8, 1.2s]\). When \( \kappa = 1 \) there is a singularity in the reconstruction algorithm which runs at FTU off-line after the experiment. This is one of the reasons why the elongation control is useful: to
7.4 Elongation regulation

![Graphs showing elongation, currents, and magnetic flux changes.](image)

Figure 7.19: Shot 32969. The elongation (upper plot), the current $i_F$ (upper-middle plot), the current $i_V$ (lower-middle plot) and $\Delta \Psi$ (lower plot) are shown.

maintain the elongation far enough from the singularity. Nevertheless this experiment shows that the elongation regulation works without problems also with $\kappa = 1$, because the reconstruction algorithm is not involved in the real-time control. In this shot the perturbations on the position due to the large and fast variations of the elongation are even more relevant and appear evident at times 0.5, 0.8 and 1.2s.

Some experiments have also been done to choose the best value for the allocator gain $\rho$. The same scenario is repeated in shots 32963, 32964 and 32966 respectively with $\rho = 1$, $\rho = 3$ and $\rho = 1.5$. Fig. 7.20
Figure 7.20: Shot 32963 with \( \rho = 1 \) (solid), shot 32964 with \( \rho = 3 \) (dotted), shot 32965 with \( \rho = 1.5 \) (dash-dotted).

shows a zoom of the time interval \([0.8, 1.2\, \text{s}]\) of the elongation \( \kappa \) and the current \( i_V \). From the figure it is clear that when too large values of the gain are used, an oscillation appears in the closed-loop, revealing some dynamics or time delays neglected in the relationship between the current \( i_F \) and the elongation \( \kappa \). The oscillations can be better appreciated looking at the current \( i_V \), which is less noisy.
Chapter 8

Saturation avoidance at JET

8.1 Introduction

The JET (Joint European Torus) tokamak, located in Culham, England, is today’s world largest fusion reactor ([38, 39, 40]). In this chapter we will apply the input allocation strategy to the problem of controlling the plasma shape in the JET tokamak [41]. In particular we will focus on the problem of saturation avoidance for the plasma shape control system. In the JET case, as will be better described in the following, we have a number of control outputs (position and shape descriptors) bigger than the number of available inputs: so we deal with an under-actuated control system. It is easy to show that this latter can be dealt with in the theoretical framework described in Chap. 4.

The results illustrated in this chapter appeared in [18], [21], [20] and [22].
8.2 The JET plasma shape controller

The JET tokamak (see Fig. 8.1) has eight poloidal field (PF) coils available for plasma shape control system. These coils are denoted by $P_1, \ldots, P_4$ and $D_1, \ldots, D_4$. The $P$-coils are connected to form five circuits [42]. The currents flowing in these circuits are denoted by $I_{PRIM}$, $I_{PAT}$, $I_{PAIM}$, $I_{PFX}$, $I_{SHP}$, whereas the currents flowing in the $D$-coils are denoted by $I_{Di}$, with $i = 1, \ldots, 4$. There are thus nine circuits available to the plasma control system. The circuit $P1E$ is used to control the plasma current, whereas the other eight circuits can be used to control the plasma shape.

The position and shape reference value for the plasma control system consists in many geometric descriptors: “gaps” (which are distances of the plasma separatrix from the first wall, namely the innermost surface of the vacuum vessel) and coordinates of the X-point and of the two strike-points. Even though direct measurements of these quantities are not available, they can be estimated from a set of magnetic measurements by the plasma shape estimation algorithm XLOC [43]. However, these geometric descriptors are larger in number than the available PF coils.

The eXtreme Shape Controller (XSC, [44]) is currently adopted at JET to accurately control highly elongated plasmas by driving the current in the Poloidal Field (PF) coils system (see Fig. 8.1). The XSC enables high accuracy control of the overall plasma boundary, specified in terms of a given number of plasma shape descriptors, i.e., gaps, strike-points and x-point (see Fig. 8.2). The XSC is based on a methodology [45] that minimizes the steady-state error in the geometrical parameters when constant references are applied. In its present implementation, anyways, the XSC does not handle current saturations in the PF coils. Indeed, each operating scenario is carefully designed in order to avoid PF currents saturation in the presence of
8.2 The JET plasma shape controller

Figure 8.1: The JET Poloidal Field coils system. The $P1$ circuit includes the elements of the central solenoid $P1EU$, $P1C$, $P1EL4$, as well as $P3MU$ and $P3ML4$. The series circuit of $P4U$ and $P4L$ is named $P4$, while the circuit that creates an imbalance current between the two coils is referred to as $IMB$. The shaping circuit $SHP$ is made of the series of $P2SU$, $P3SU$, $P2SL$, and $P3SL$. The central part of the central solenoid contains an additional circuit named $PFX$. Finally the four divertor coils ($D1$ to $D4$) are driven separately each by one power supply.

the envisaged disturbances (see [?]).

For this reason a dynamic input allocator, based on the results on under-actuated plants exposed in Chap. 4 has been implemented and integrated with the XSC to manage current limit avoidance. The input allocator is used to modify the output of the XSC in order to avoid current saturations in the eight active circuits. To this aim, the current $I_{PRIM}$ in the $P1$ circuit is excluded from the allocable
Saturation avoidance at JET

Figure 8.2: Plasma boundary descriptors. This figure shows the strike points and the X point, together with gaps typically controlled on the JET tokamak.

ones following the approach in Remark 4.3, since this circuit is used to control the plasma current. The cost function $J$ the allocator tries to minimize has the form of 4.30. The allocator was implemented in such a way that it is possible to choose between the output driven and the error driven configurations. When the JET tokamak plant is considered, the control input $u$ is represented by the 8 currents flowing in the PF coils devoted to the plasma shape control, while the controlled outputs $y$ are represented by the $p$ plasma shape descriptors, where typically $p \geq 30$. Furthermore, the disturbance vector $d$ holds the poloidal beta $\beta_p$ and the internal inductance $l_i$ variations, $^1$ together with the variation of the plasma current $I_p$.

To evaluate the performance of the new allocator block, it was integrated within the XSC and many simulations were performed with a linear model of the plasma (the CREATE-L model, see [47]).

$^1 \beta_p$ and $l_i$ are measures of the plasma internal distributions of pressure and current, respectively (see also [46]).
8.3 Open-loop simulations

A series of open loop simulations of the output driven allocator with constant input (i.e. constant controller output $y_c$) is carried out here in order to show how the allocator works and also how it can be used to adjust the desired scenario before the experiment. In particular an equilibrium configuration has been considered in which the desired current $I_{D1}$ in the D1 coil is at the upper saturation limit corresponding to 0 A (note that this implies that the blue bar is not visible in the $I_{D1}$ histograms of Fig. 8.3).

In all the tests the expression (4.30) is used for the cost function $J$, with the following parameters selection:

$$\rho = 500000, \quad (8.1a)$$

$$K = I, \quad (8.1b)$$

$$a_i = b_i = 1, \quad (8.1c)$$

$$\bar{a}_i = a_i = 0.1, \quad (8.1d)$$

and $u_i$, $\bar{u}_i$ corresponding to 35% of the actual currents’ ranges.

In Fig. 8.4 the matrix $B_0$ is chosen equal to the identity matrix in order to exploit all the available input directions and the following output penalties are chosen: $\bar{b}_i = b_i = 3 \cdot 10^6$. The steady-state input allocation is shown at the top plot of Fig. 8.3 where it appears that the input allocator moves the inputs away from the saturation limits. In particular, notice that the allocated current in $D1$ coil moved away from its upper limit.

As a second test, in Fig. 8.5 we leave all the parameters unchanged except for the output penalties which are increased by a factor 100, that is $\bar{b}_i = b_i = 3 \cdot 10^8$, so that the output deviations will be penalized much more by the allocator. As expected, the steady-state allocation induces a significantly smaller shape deformation (compare Fig. 8.5 to Fig. 8.4). From the middle plot of Fig. 8.3 it appears that the output penalty doesn’t allow the allocator to move $I_{D1}$ as much as in
Figure 8.3: Input ranges (red), controller output (blue), steady-state allocated input (green) for the three cases shown in Figs 8.4-8.6. Note that the $I_{D1}$ current has no blue bar because its value before allocation is 0, while after allocation it is moved away from saturation.
the previous case. However, the allocator still manages to move the current away from the saturation limit.

Finally, following Remark 4.3, in Fig. 8.6 the $B_0$ matrix is changed in order to show the possible use of (4.23) and (4.24) to block some of the input and output variables. It turns out that reducing the number of input directions available for the allocation, some particular constraints are respected, but generally the maximum deviations from the setpoint can increase. All the other parameters have the same values of the first test. In Fig. 8.6 the X-point is blocked (namely, $CV - ZX$ and $CV - RX$ are fixed), the $ROG$ output is blocked and two of the four strike points variables (namely, $ZSOGB$ and $RSOGB$) are blocked. Moreover, the allocator is required not to modify the current in the $IP4T$ coil. The arising steady-state input allocation is
Figure 8.5: Plasma shape (left) and output deviations (right) with no fixed outputs or inputs and the selection and the selection $\bar{b}_i = \hat{b}_i = 3 \cdot 10^8$.

shown at the bottom plot of Fig. 8.3 where it is possible to appreciate a significant effectiveness in moving $I_{D1}$ away from saturation, while satisfying all the imposed constraints.
Figure 8.6: Plasma shape (left) and output deviations (right) with five fixed outputs (namely \( CV - RX \), \( CV - ZX \), \( ZSOGB \), \( RSIGB \) and \( RSOGB \), i.e. X-point and strike points) and one fixed input (\( IP4T \) current).

### 8.4 Feasible regions

In this section we want to show the potential benefits arising from the use of the input allocation in error driven configuration, represented in Fig. 4.17, with respect to the original one which penalizes output deviations, reported in Fig. 4.16. To this aim, we consider the analysis performed to estimate the enlargement of the operating space in terms of maximum rejectable constant disturbances \( \Delta \beta_p \) and \( \Delta l_i \), due to the adoption of an input allocation scheme.

In particular, the maximum tolerable errors for the set of plasma shape descriptors shown in Table 8.1 have been set, and the satura-
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<th>$\Delta_{\text{max}}$ cm</th>
<th>$\Delta_{\text{min}}$ cm</th>
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<tr>
<td>Radial Outer Gap</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>Radial Inner Gap</td>
<td>5</td>
<td>-5</td>
</tr>
<tr>
<td>Top Gap</td>
<td>10</td>
<td>-10</td>
</tr>
<tr>
<td>Outer Strike-point</td>
<td>2</td>
<td>-5</td>
</tr>
<tr>
<td>Inner Strike-point</td>
<td>10</td>
<td>-4</td>
</tr>
</tbody>
</table>

Table 8.1: Maximum tolerable error for a set of plasma shape geometrical descriptors used for the steady-state analysis reported in Section 8.2.

...tion constraints for the PF currents have been considered. Given these constraints, the maximum $\Delta \beta_p$, $\Delta l_i$ obtainable by the XSC with and without the input allocation, in both the possible configurations, have been computed by means of a steady-state analysis, i.e., by using a static model for both the plant and the XSC. As an example of steady-state analysis that has been carried out to assess the performance of the input allocator, the JET pulse # 78668 at $t = 13.4$ s is considered. The equilibrium values of the plasma current, poloidal beta and internal inductance are $I_{p_{eq}} = 3.4$ $MA$, $\beta_{p_{eq}} = 0.29$, and $l_{i_{eq}} = 0.91$, respectively. The results are reported in Fig. 8.7. In particular, three different operating spaces are shown:

1. the yellow area which corresponds to the operating space obtained by using the XSC alone;

2. the blue area is the operating space obtained by using the XSC with the output driven allocator of Fig. 4.16;

3. the green area is the operating space obtained by using the XSC with the error driven allocator of Fig. 4.17.

Note that there is only a minor enlargement of the operating space from case (1) to case (2), while a big advantage is obtained by using the scheme in Fig. ??.
8.5 Closed-loop simulations

In this section we illustrate the effectiveness of the allocator in the closed-loop configuration described and analyzed in Chap. 4. The allocator is used to prevent the steady-state saturation of the currents in the poloidal field (PF) coils used to control the plasma shape in the JET tokamak, so that the mean square error of the shape descriptors is reduced as compared to the non-allocated case.

In a first simulation (Test Case A) we consider a shape tracking problem where the shape reference moves from a constant shape to a new. In this case the allocator in output driven configuration is used. Motivated by the results in Sec. 8.4, in the following two simulations the error driven allocator is used. In particular, the following two cases are

Figure 8.7: Operative space in terms of $\beta_p$ and $l_i$ when the plasma current is equal to 3.4 $MA$. This analysis has been performed assuming a static model for both the plant and the controller. The $\beta_p$ range has been set equal to $[0.1, 2.3]$, while the $l_i$ is $[0.4, 1.5]$.

linear model, hence the real advantage may be less than the expected one.
considered in order to show the effectiveness of this approach: in the first case (Test Case B) the input allocation allows to operate a given scenario at higher plasma current without saturating the currents in the PF coils. In the second case (Test Case C) the input allocation is adopted to move the PF currents far from their saturation limits, in order to operate a given scenario in a safer way.

For all the dynamic simulations the allocator has been integrated into the XSCSIMULATOR described in [48].

8.5.1 Test Case A - Changing shape reference

In a first example we consider a shape tracking problem using the eXtreme Shape Controller (XSC, see [41]) where the shape reference moves from the V5,3M5,HT3 configuration to a new shape which has both the strike-points moved outward. In particular, the shape reference is constant until time $t_1 = 61\ s$ and is constant again after $t_2 = 61.5\ s$. In the interval $[t_1, t_2]$ the shape references ramp up from the old configuration to the new one. In this simulation the allocator works in output driven configuration. Despite the geometrical aspects, what is relevant in this simulation is that, without the allocator, the current in the $D1$ circuit is permanently saturated at 0. This appears from Fig. 8.8 where the lower plot shows the current reference before saturation. The controller commands 100 $KA$ of current, which is way beyond any reasonable current demand. Indeed, for the considered JET configuration, the current in $D1$ is constrained in the range $[-19\ kA, 0\ kA]$. From the red curve of Fig. 8.9 a small error appears in the shape before $t_1$ (because the $D1$ current reference is only slightly into saturation), the error then increases with a peak at the end of the reference ramp-up phase and settles to a steady-state error value, caused by the deep input saturation.

When the allocator is used (blue curves of Fig. 8.8), the current in $D1$ is moved away from the saturation before $t_1$ at the price of an increased shape error (see the blue curve in Fig. 8.9). However, at
8.5 Closed-loop simulations

$t = 61 \text{ s}$, when the reference starts changing, the 1.3 kA input range available on the $D1$ circuit allows for reduction of the mean square error peak. Finally, at the steady-state after $t \approx 62$, the allocator is able to 1) induce a current in $D1$ which still guarantees a margin of 650 A (see the blue curve in the upper plot of Fig. 8.8); 2) reduce the steady-state mean square error on the plasma shape.

8.5.2 Test Case B - Increasing the plasma current

In this case we have considered the JET pulse # 74177 at $t = 8.8 \text{ s}$. For this equilibrium we have $I_{p,eq} = 4 \text{ MA}$, $\beta_{p,eq} = 0.16$, and $l_{i,eq} =$
Figure 8.9: Test Case A. Shape tracking. Mean square error on the controlled shape descriptors.

Figure 8.10: Test Case B. Plasma current variation. The plasma current is increased up to 4.5 MA.

0.75. In this operative scenario plasma current is limited by the current in the $D2$ circuit, which is close to its saturation limit. Indeed, the equilibrium value of the current in the $D2$ circuit is equal to $I_{D_{2eq}} = -33.5 \, kA$, which is close to its lower saturation limit ($-37 \, kA$). The allocator in error-driven mode has been used to limit $I_{D2}$ in the range $[-31.45, -5.5] \, kA$, and to increase the plasma cur-
8.5 Closed-loop simulations

Figure 8.11: Test Case B. Current in the PF circuits when $I_p$ is increased up to 4.5 MA and the allocator in error-driven mode is switched on. Note that the current in $D2$ is kept in the range $[-31.45, -5.5]$ kA.

rent up to 4.5 MA (see Fig. 8.10). The allocator keeps the current in $D2$ within its new limits, while the plasma shape error increases in the upper zone (see Figs. 8.11 and 8.12). It should be recalled that the plasma current variation is seen as a disturbance for the shape control.

8.5.3 Test Case C - Keeping PF coil currents far from saturation

The JET pulse # 78668 at $t = 13.4$ s, already used for the steady-state analysis, is considered in this test case. The equilibrium values of the plasma current, poloidal beta and internal inductance are $I_{p_{eq}} = 3.4$ MA, $\beta_{p_{eq}} = 0.29$, and $l_{i_{eq}} = 0.91$. In this configuration the currents in two control circuits, namely the $PFX$ and the $SHP$ circuits, are very close to their limits. Indeed, their equilibrium values are equal to $I_{PFX} = 29.2$ kA and $I_{SHP} = 32.7$ kA, while their upper saturation limits are equal to $I_{PFX_{max}} = 32.9$ kA and $I_{SHP_{max}} = 38$ kA. In this case, new ranges are set for these
two currents, then the allocator in error-driven mode is used to move them far from their new limits. In particular, we let PFX to range in the interval $[4.9, 28] \text{ kA}$, while the $SHP$ current is limited in the range $[9.5, 28.5] \text{ kA}$. Fig. 8.14 shows the behavior of the currents in the PF coils. The plasma shape achieved in steady-state is shown in Fig. 8.13 (the shape reference has been set equal to the experimental shape at $t = 13.4 \text{ s}$).
Figure 8.13: Test Case C. Plasma shape when the allocator is used to move both the PFX and the SHP currents far from their saturation limits. The plasma shape at $t = 13.4$ s has been set as reference for the XSC. The shape reference is the red trace, while the green trace is the simulated shape.
Figure 8.14: *Test Case C*. PF currents when the allocator is used to move both *PFX* and the *SHP* currents far from their saturation limits. Note that both the *PFX* and the *SHP* currents are kept equal to their new upper bounds, in order to minimize the error on the plasma shape tracking. It should also be noticed that the allocator *slows down* its reaction, once the PFX and SHP currents are in far from their limits.
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