Stochastic Bounds on Execution Times of Parallel Computations

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Abstract

In this paper we obtain stochastic bounds on execution times of parallel computations assuming ideal conditions for shared resources. A parallel computation is modelled as a task system with precedence constraints expressed as a Directed Acyclic Graph (DAG). The task execution times are assumed independent random variables. The performance measure we consider is the overall execution time of the computation. To obtain upper bounds on this measure, we apply stochastic ordering and stochastic comparison techniques.

Keywords: parallel computations, task graph, performance evalutation, stochastic ordering, stochastic bounds.

I Introduction

Parallel computations are usually structured as a set of tasks executing on concurrent systems and can be well described by a task graph, where nodes represent tasks and directed arcs represent synchronization constraints between tasks. Assuming ideal conditions for shared resources the overall completation time T_G of the graph is given by the duration of the longest path; hence, said π_j the duration of path p_j, we have $T_G = \max \pi_j$ with distribution $F(t) = P\left[\max \pi_j \le t\right]$.

Approximating or bounding methods has been widely investigated (e.g. PERT networks [Kl71]) in order to make such analysis tractable. In this paper we apply stochastic bound method to obtain upper bounds for F(*t*), based on *stochastic ordering* for random variables [St83]. The proposed approach results computationally efficient, and in its *naive* form it follows that of Kleindorfer [Kl71].

II The Model

We model parallel computations as Direct Acyclic Graph (DAG), called *task graph*, where the arcs represent precedence constraints between tasks modelled by nodes. We use the following notations:

- G=(N,A,S,T) is a directed and acyclic graph where:
- N is the task set {1,2,...,*n*}
- A is a subset of N×N, said set of arcs
- S is the set of task with no predecessors (S⊆N)
- T is the set of task with no successors (T⊆N)

The random variables S_i describing the task execution times are assumed independent with known distribution $C_i(t)$, $i \in N$.

III Upper Bound

If T_i denotes the completion time of the task i (i=1,2,...,n), we have the following recursive equation:

$$T_i = \max_{j \in p(i)} \{T_j\} + S_i$$

$$T_G = \max_{j \in T} \{T_j\}$$

where p(i) is the set of immediate predecessors of i, $i \in N$, while s(i) is the set of immediate successors.

By assuming independent execution times, we have the following basic result:

Theorem 1. Let $\mathbf{T} = \{T_1, \ldots, T_n\}$ be the vector of the completion times. The random variables T_i, $i \in N$, are associated.

The upper bounds for task completion and overall computation times are given by the following set of distributions:

$$\hat{F}_i(t) = F_i(t) = C_i(t) \qquad i \in S$$

$$\hat{F}_i(t) = \left[\prod_{j \in p(i)} \hat{F}_j(t)\right] \otimes C_i(t) \qquad i \notin S$$

$$\hat{F}(t) = \prod_{j \in T} \hat{F}_j(t)$$

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Theorem 2. The distribution F_i, $i \in N$, and F are stochastically smaller than the distribution $\hat{F}_i$ and $\hat{F}$, respectively:

$$\hat{F}_i \geq_{st} F_i \qquad i \in N, \qquad \hat{F} \geq_{st} F.$$

Now we show that it is possible to obtain tighter bounds by *reducing* series-parallel subgraphs contained in a graph G (i.e., by carrying out all possible *graph reduction* operation). A *graph reduction* operation is possible if [Do85]:

1. there is at least one node $i \in$N with just one outgoing arc (i,j), and this is the only arc incident on j. The nodes i and j are said to be in series. It is possible to *reduce* the graph by substituting a single node to the nodes i and j, the distribution of which is obtained by convoluting the distribution of the two nodes in series.
2. there are at least two nodes i and j such that p(i)=p(j) and s(i)=s(j). The nodes i and j are in parallel. It is possible to *reduce* the graph, by substituting to the nodes i and j a single node, the distribution of which is obtained by multiplying the distribution of the two nodes in parallel.

The following theorem holds:

Theorem 3. If G_R denotes the graph obtained by *reducing* series-parallel subgraphs in a given graph G, the following inequality holds (with $\hat{F}_R$ the distribution of G_R):

$$\hat{F}_R \leq_{st} \hat{F}.$$

Proof. (full details are in [Lop93]).

Reducing series-parallel subgraphs consists on carrying out all possible *graph reduction* operations; then we have to show only that the graph G'=(N',A',S',T'), obtained after a single *graph reduction* operation (for space lack, here we consider only the *parallel graph reduction* case) from the graph G, is such that : $\hat{F}' \leq_{st} \hat{F}$ (with $\hat{F}'$ the distribution of $\hat{G}'$).

Let us call i and j the two nodes in parallel in G, and i-p-j the reduced node in G'. The expressions for $\hat{F}_i$ and $\hat{F}_j$ are the following (remember that p(i)=p(j)=p'(i-p-j) and s(i)=s(j)=s'(i-p-j)):

$$\hat{F}_i = \left[\prod_{l \in p(i)} \hat{F}_l\right] \otimes C_i, \quad \hat{F}_j = \left[\prod_{l \in p(i)} \hat{F}_l\right] \otimes C_j,$$

whereas the expression for $\hat{F}_{i-p-j}$ in G' is:

$$\hat{F}_{i-p-j} = \left[\prod_{l \in p(i)} \hat{F}_l\right] \otimes (C_i C_j)$$

The following result holds (a simple proof is given, for example, in [Do85]): $\hat{F}_i \hat{F}_j \geq_{st} \hat{F}_{i-p-j}$.

Now we demonstrate that $\hat{F}' \leq_{st} \hat{F}$.

The proof is based on induction, showing that $\hat{F}_m' \leq_{st} \hat{F}_m$ for $m \in$N'=[N-{i,j}]$\cup${ i-par-j}.

For $m \in$N', we have (after having proved that $\hat{F}_k' \leq_{st} \hat{F}_k$ $k \in$p(m), $k \neq i$-p-j), two cases:

- i-p-$l \notin$ p(m).

$$\hat{F}_m' = \prod_{k \in p'(m)} \hat{F}_k' \otimes C_m \leq_{st} \prod_{k \in p(m)} \hat{F}_k \otimes C_m = \hat{F}_m$$

- i-p-$l \in$ p(m) (recall that if i-p-$j \in$p'(m) in G', i,j $\in$p(m) in G).

$$\hat{F}_m' = \left(\prod_{k \in p'(m)-\{i-p-j\}} \hat{F}_k'\right) \hat{F}_{i-p-j} \otimes C_m \leq_{st} \left(\prod_{k \in p(m)-\{i,j\}} \hat{F}_k\right) \hat{F}_i \hat{F}_j \otimes C_m = \hat{F}_m$$

IV Conclusions

In this paper we have introduced a computationally efficient approach to obtain stochastic upper bounds for the overall execution time of parallel computations modeled by DAG ($O(n)$ convolution operation and $O(n^2)$ multiplication are needed for the evaluation of $\hat{F}$). The method uses both stochastic order for random variables and associated (positevely correlated) random variables. Moreover we propose a further improvement of these bounds by series-parallel subgraphs reductions.

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